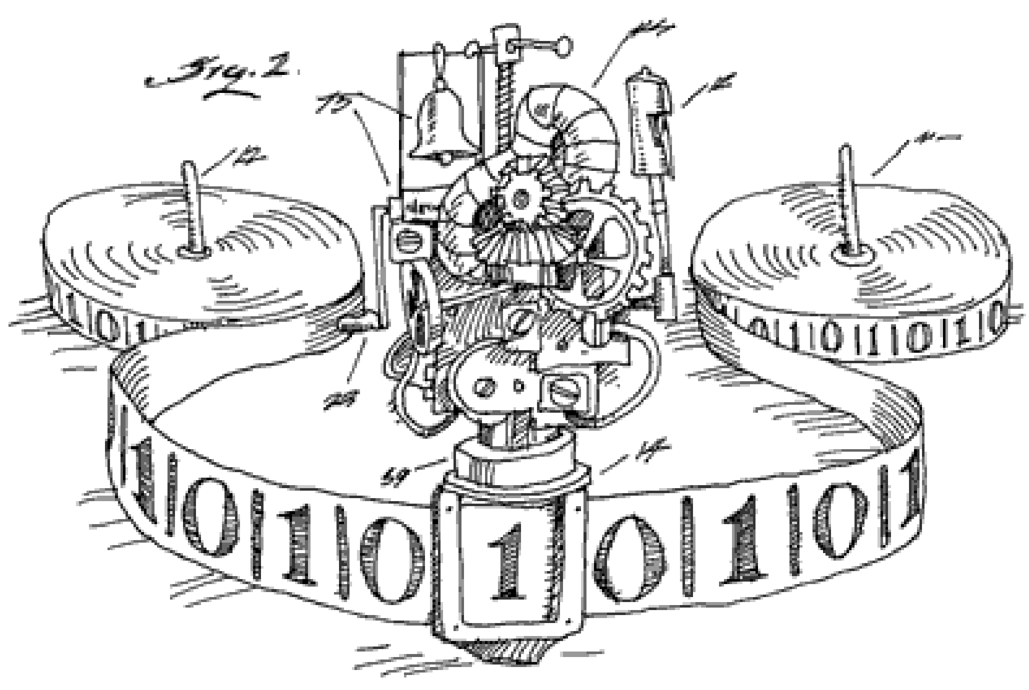
The Math Behind the Machine

Grant Schoenebeck



Attribution

- These slides were prepared for the New Jersey Governor’s School course “The Math Behind the Machine” taught in the summer of 2012 by Grant Schoenebeck
- The first half of these slides were copied or modified from a previous years’ courses given by Troy Lee in 2010, and the second half are based closely upon Chapter 10 of *Networks, Crowds, and Markets* by Kleinberg and Easley

Questions?

This course seeks to equip the participants to think like and ask questions like a theoretical computer scientist and to allow the participants experience the beauty and power of mathematics while discussing such questions as: Are there well-posed questions that no conceivable computer can solve? What types of problems are computers good at and which types can they not solve efficiently?   When can flipping coins (randomness) actually help computers? How can we measure the content of data? How can we judge how good an algorithm is?

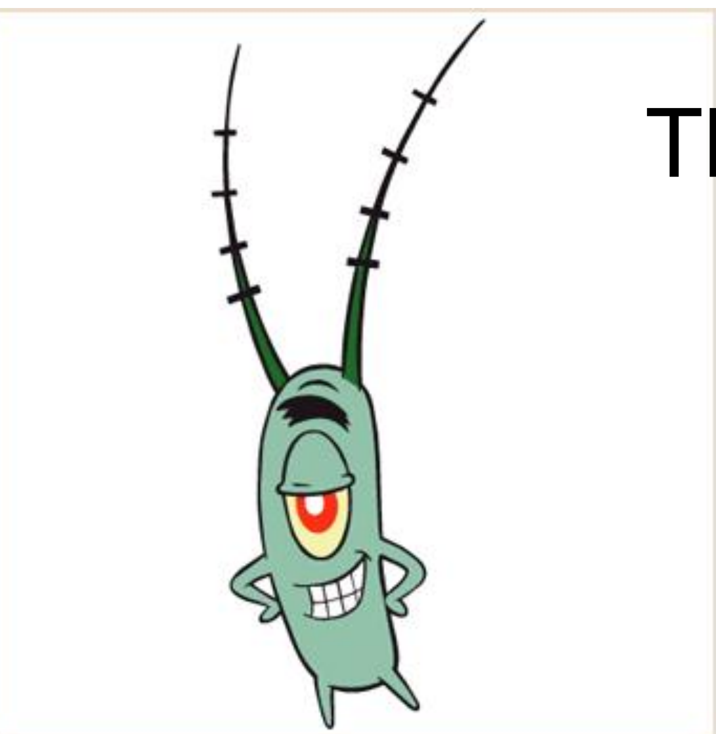
Why are you here?

Goals of Today

- Ask lots of questions
- Methodology: Example of how rigorous mathematical arguments can give insights

Stable Marriage

The mathematics of 1950's dating



Dating Scenario

A romantic candlelit dinner... with n boys and n girls.

Boys want to pair with girls, and vice versa.

What matching metric should be used to maximize the happiness of the couples?

Rogue Couples




Alice prefers Bob to her current match
Bob prefers Alice to his current match.

Can we match everyone without any rogue couples?

Modeling

Assumptions:

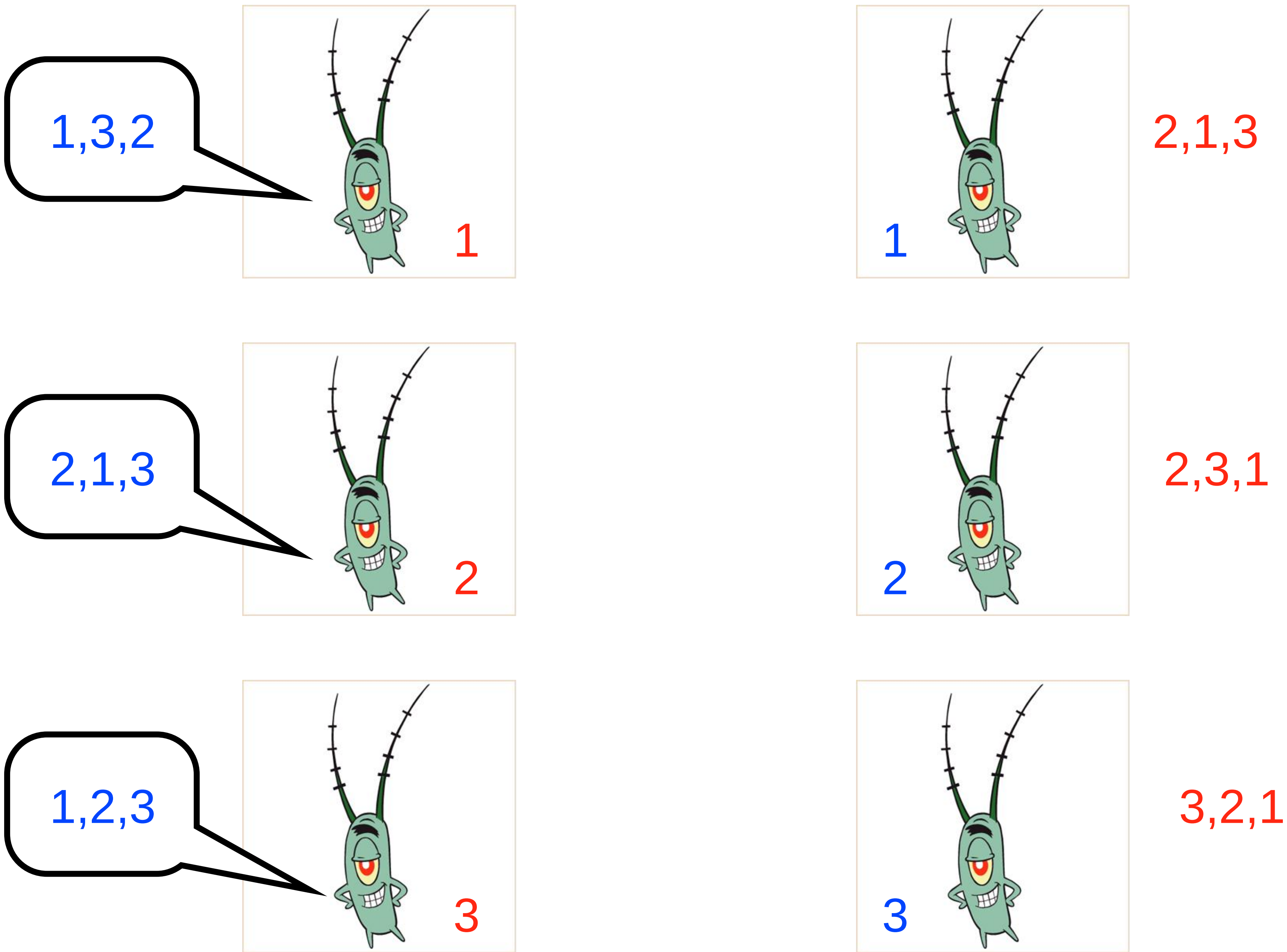
- **Two** sided matching
- **Stability** is goal
- Each side **ranks** the other

Questions:

- Does there **always exist** a stable matching?
- Can we find it **efficiently**?
- Is it **unique**? If not what **kinds** of solutions exists.
- How much **utility** is sacrificed for stability?
- Is two-sided assumption **necessary**?

Stable Pairing?

A pairing is called stable if it contains no rogue couples.



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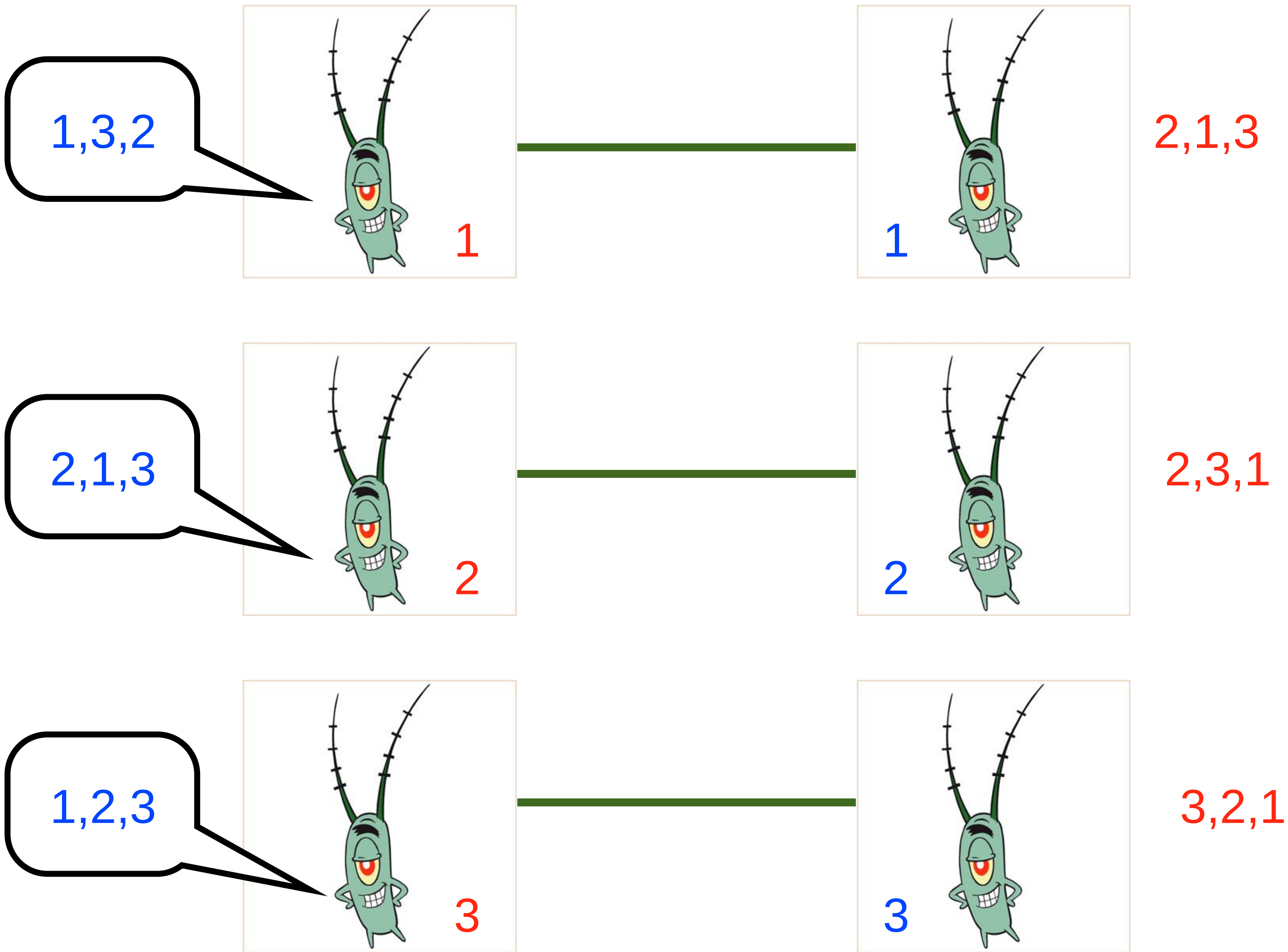

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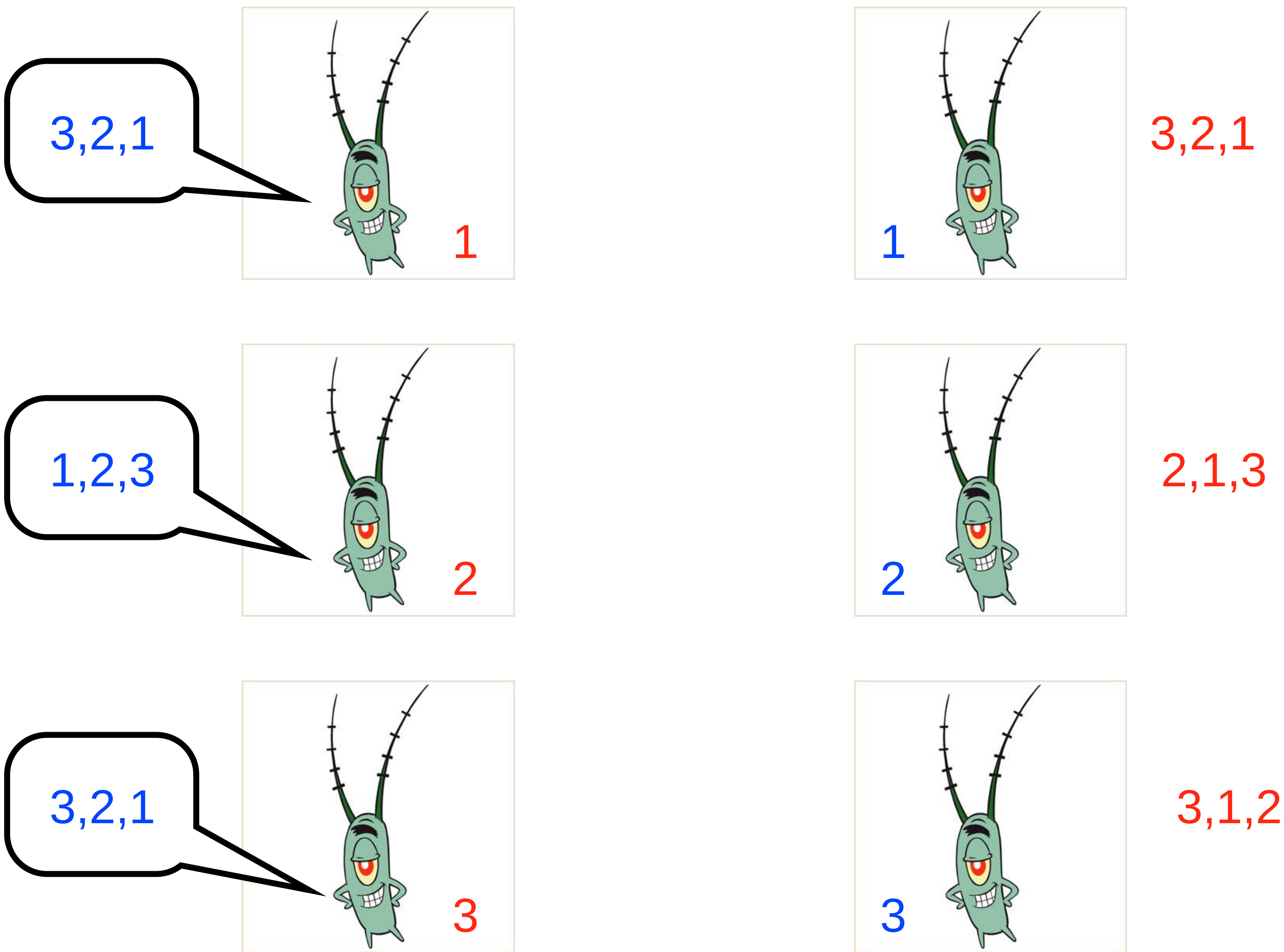

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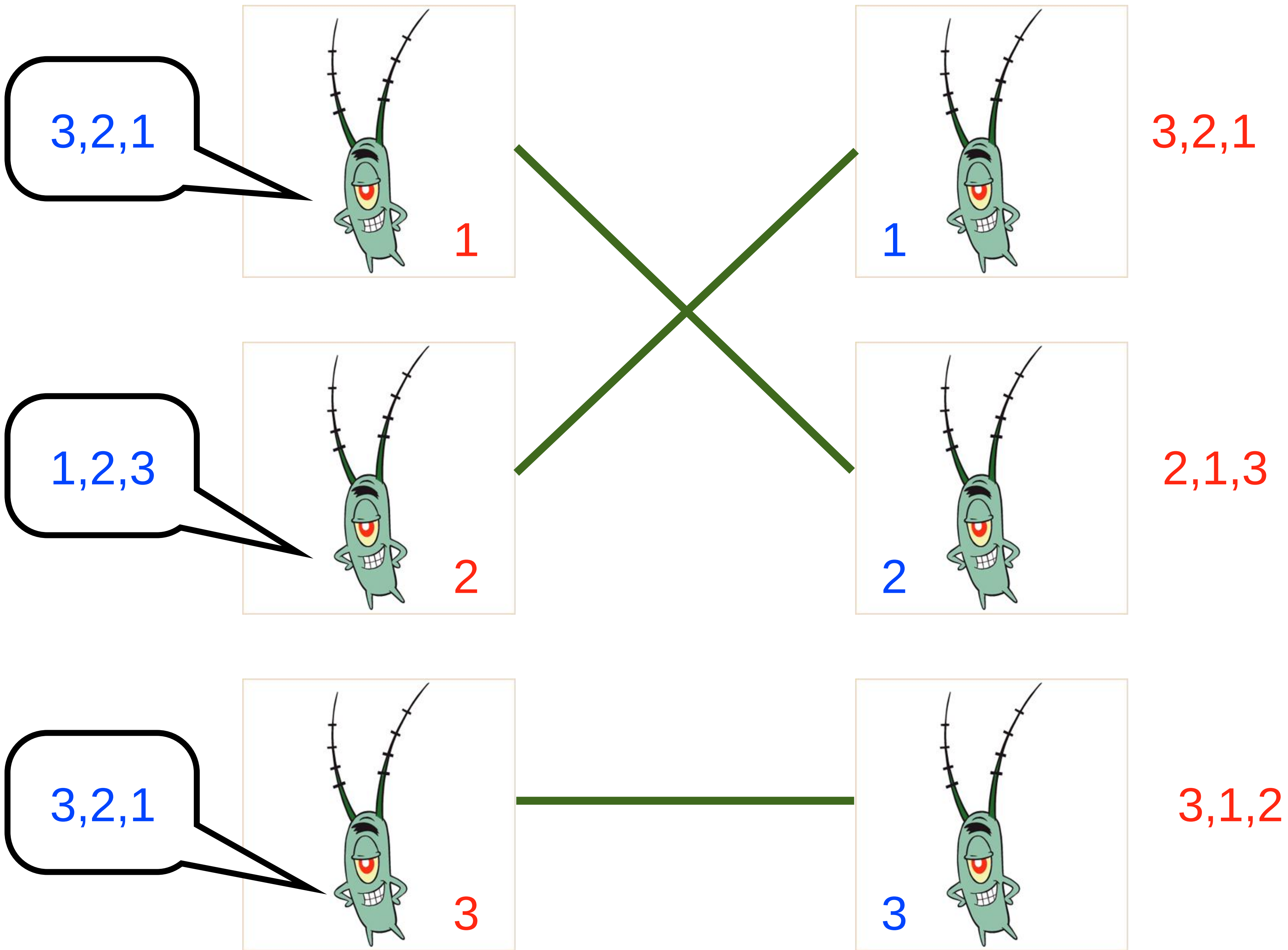
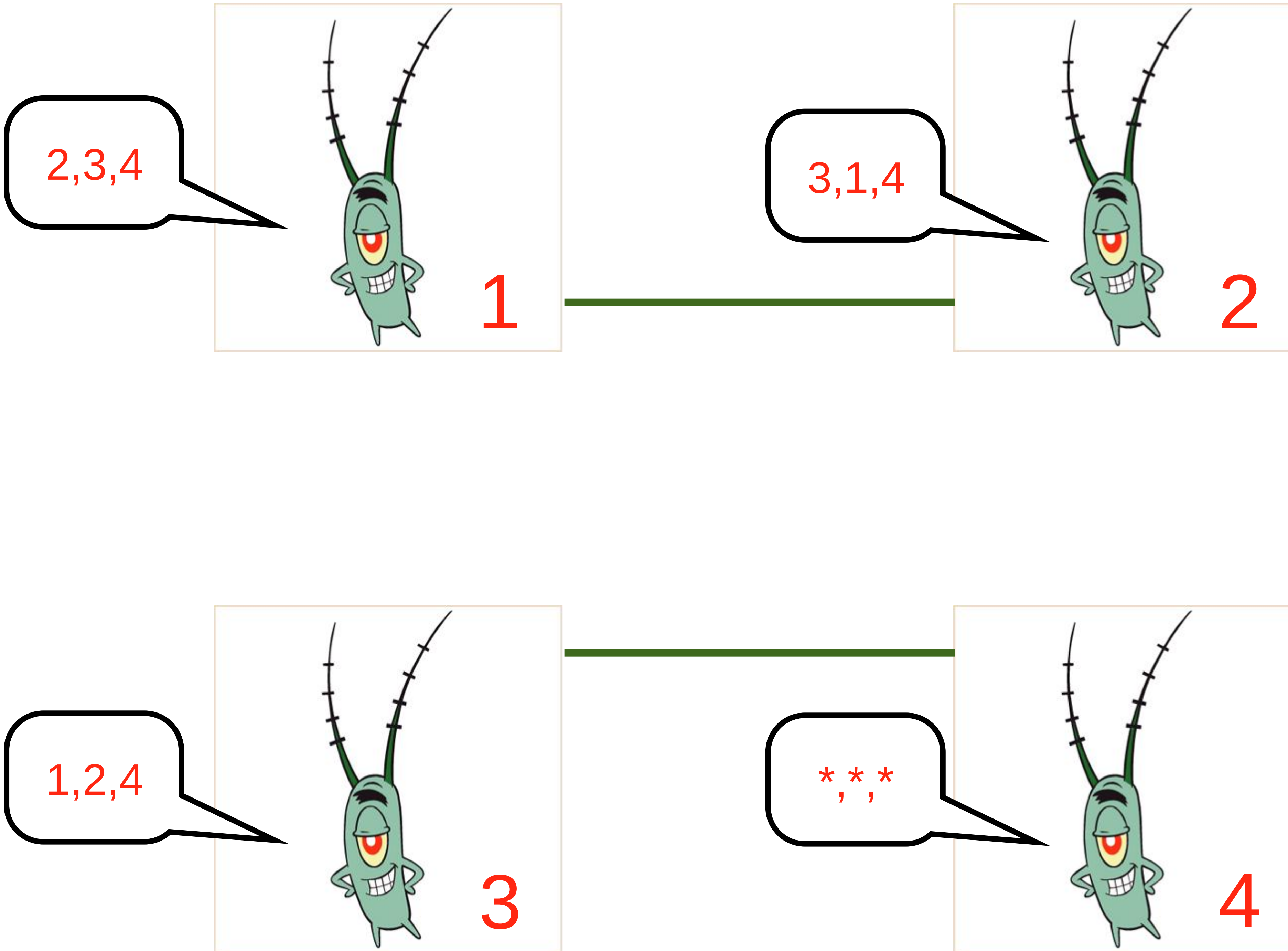


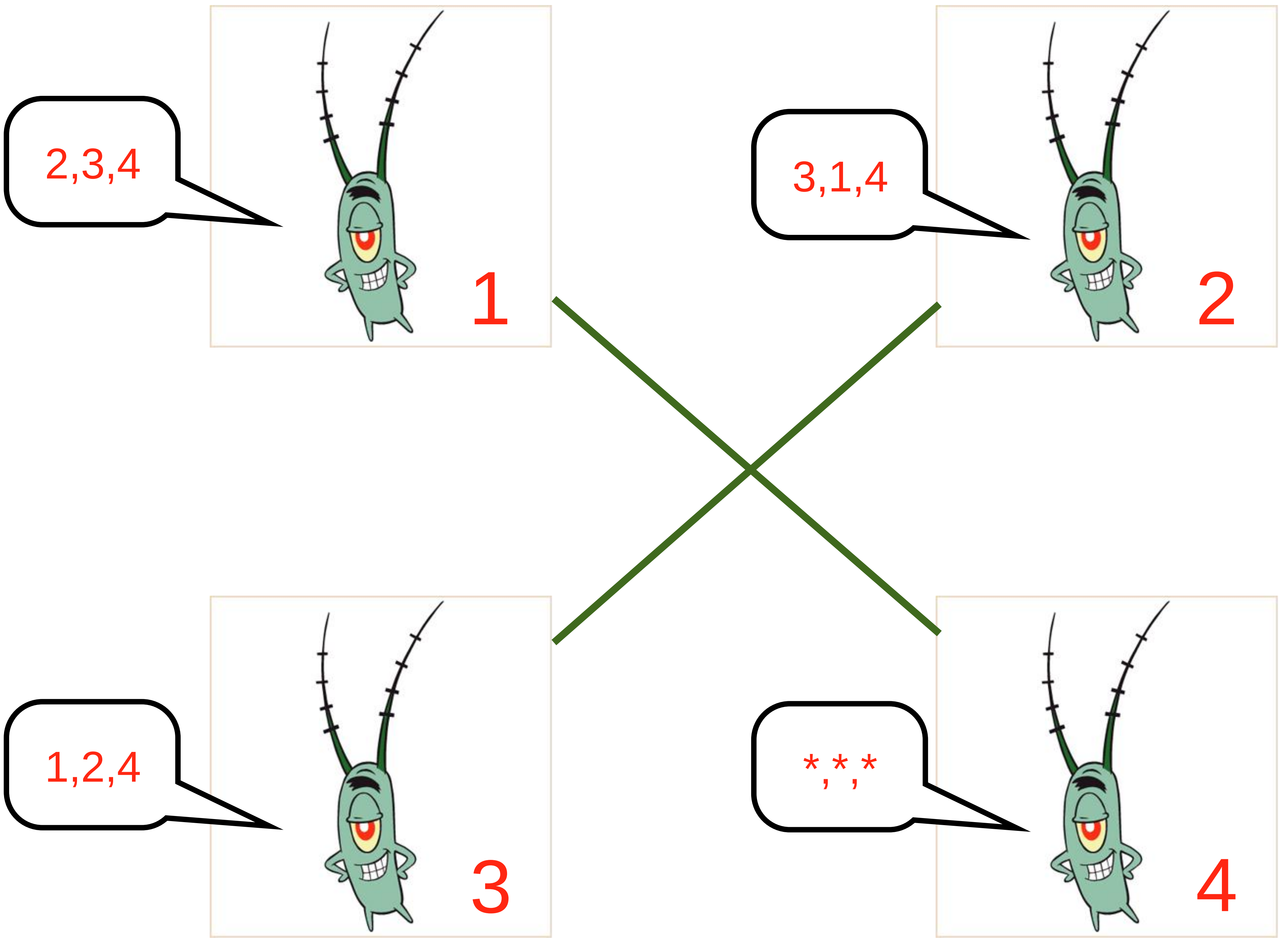
Bisexual dating

What if there were only one sex?

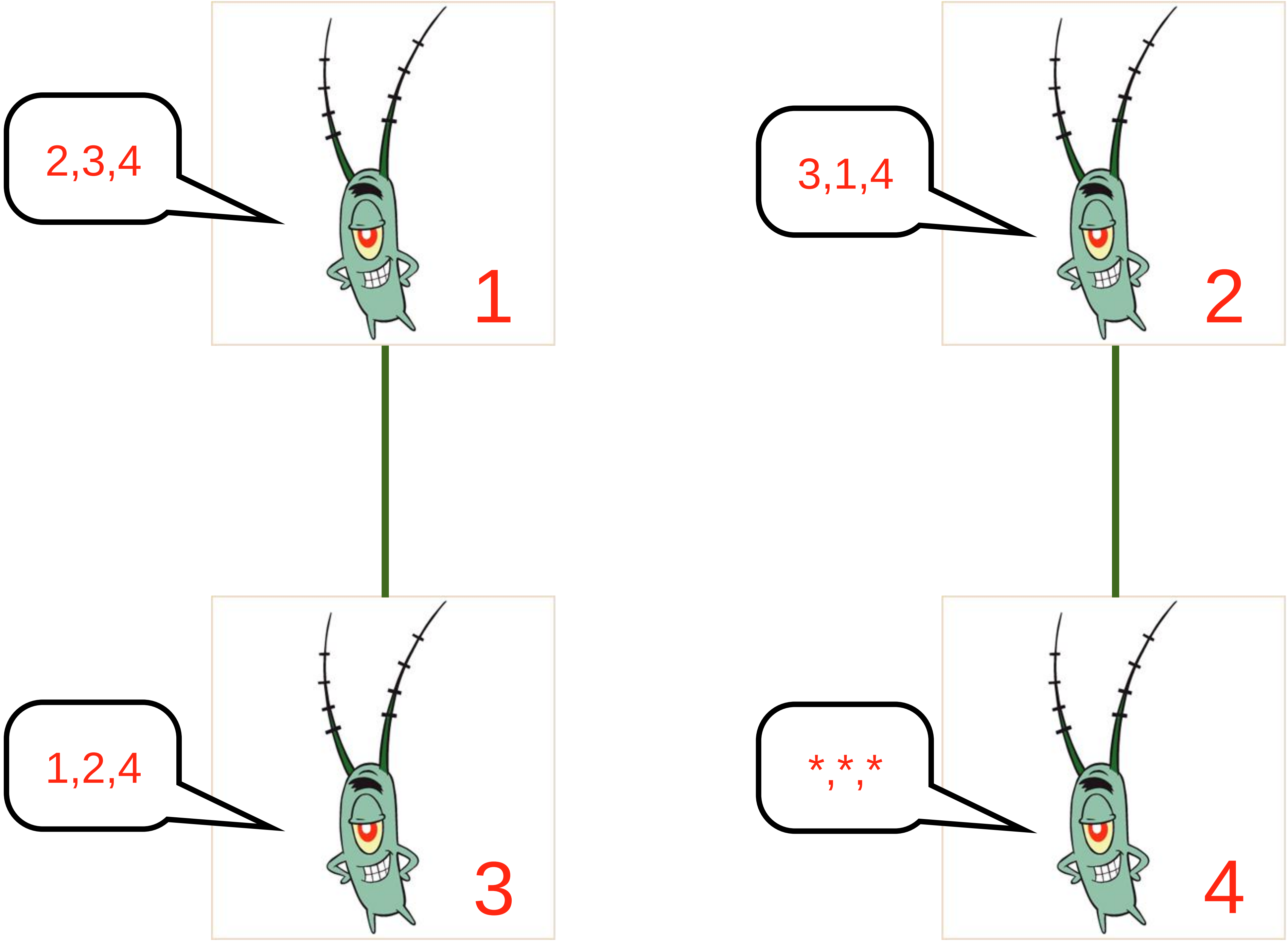
Bisexual dating



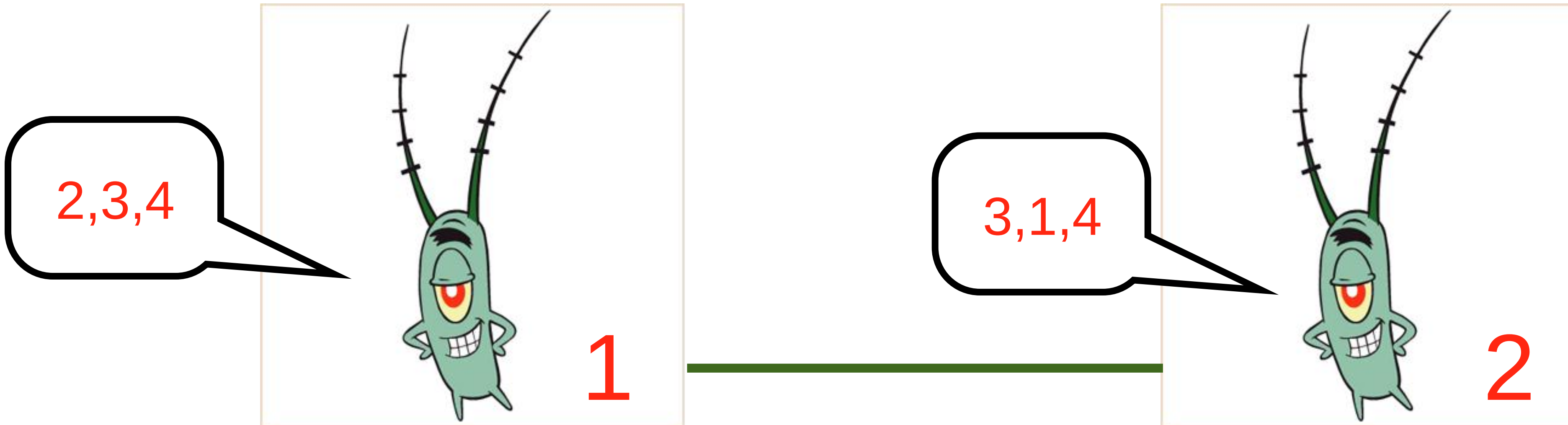
Bisexual dating



Bisexual dating



Bisexual dating

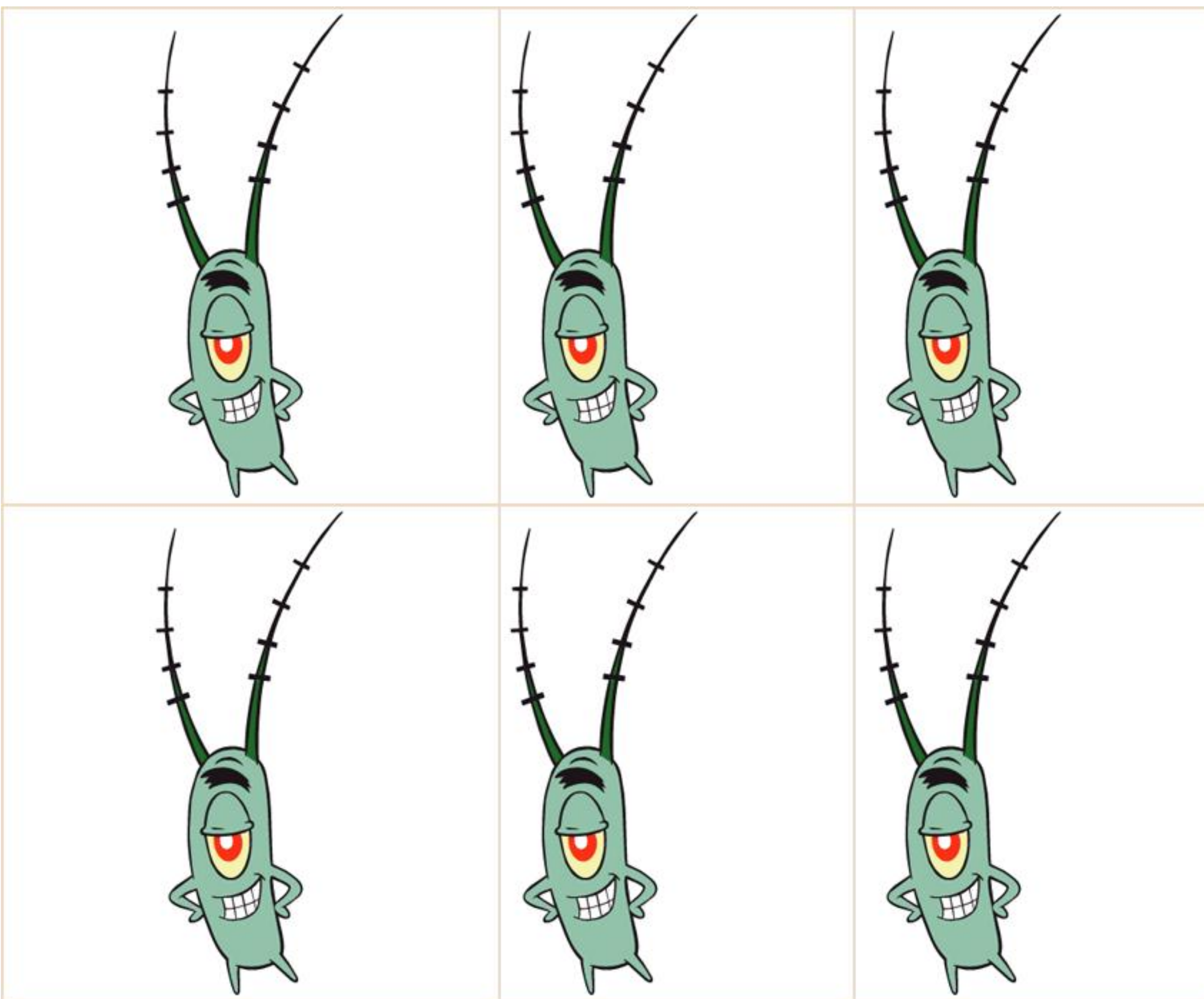


No stable pairing!

Insights?

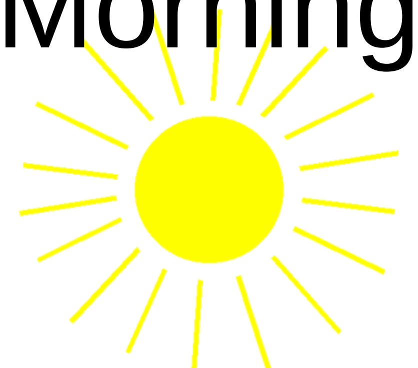
- Seems like it might be true.
- How did you find matchings in example?
- Any proof will have to use the difference between boys and girls (must break down in the bisexual case)

“Traditional” Marriage Algorithm



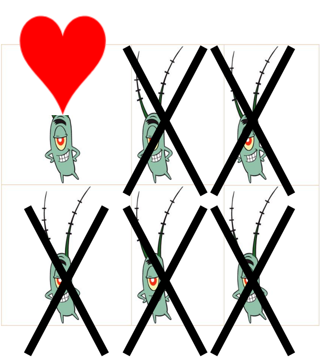
“Traditional” Marriage Algorithm

Morning:



Every girl stands on her balcony.
Every boy proposes under the balcony of the top girl remaining on his list.

Afternoon:



Girls to today’s best suitor “Maybe, come back tomorrow.”
Has boy “on a string.”
To others “I will never marry you.”

Evening:

Rejected boys cross girl off their list.

If no boy was rejected, each girl marries boy on her string.

“Traditional” Marriage Algorithm

- What now? Does this algorithm always produce a stable pairing?
- Will it always terminate?
- We’ll show

1. TMA always terminates
2. If TMA terminates, then it produces stable matching

Improvement Lemma

If a girl gets a boy b “on a string,” then in all later days she will have a boy at least as preferable as b on a string (or for a husband).

If girl does not ever reject b, he will stay on her string.

A girl will only reject b if a preferable boy b* proposes.

A girl will only reject b* if a preferable boy b** proposes.

And so on...Formally, can show lemma by induction.

Improvement lemma

A corollary of the improvement lemma:

Each girl marries her absolute favorite of the boys who visit her during the algorithm.

What about the boys?

Lemma: No boy is rejected by all the girls.

Proof by contradiction: Suppose b is rejected by all the girls.

At that point each girl will have a suitor. This follows from the improvement lemma: any boy is preferable to no boy.

As there are n girls, this gives n boys with b not among them. Contradiction!

Desperation Lemma

If a boy propose to a girl, then in all later days he proposes to girls that are no more preferable.

TMA will terminate

And it will take at most n^2 days!

Consider the master list of all boys preferences.

This has n^2 entries.

If no boy is rejected, algorithm terminates.
Otherwise, some boy is rejected and
number of entries goes down.

Note that once boy has just one girl on his list,
she must eventually marry him.

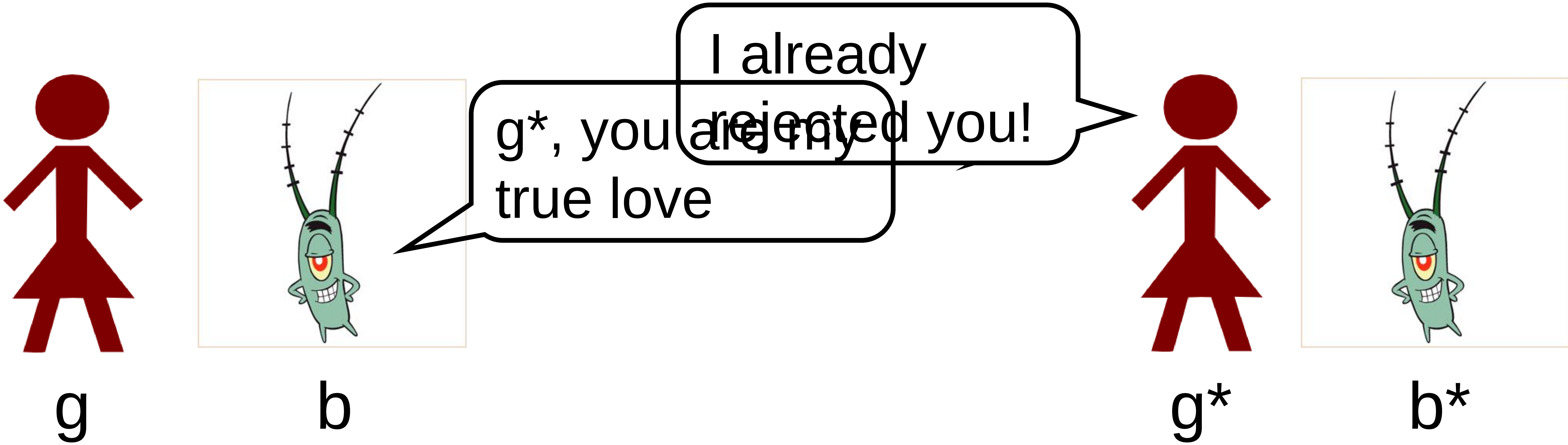
Recap

We know:

- TMA will halt and produce a pairing.
- running time of the TMA.

Will it produce a stable pairing?

The pairing is stable

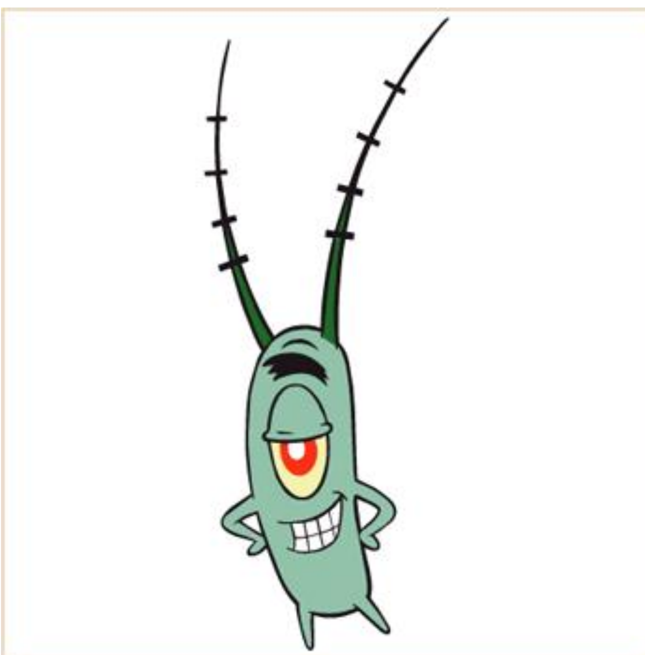


If boy b prefers g* to g, then he proposed to her first, and was at some point rejected.

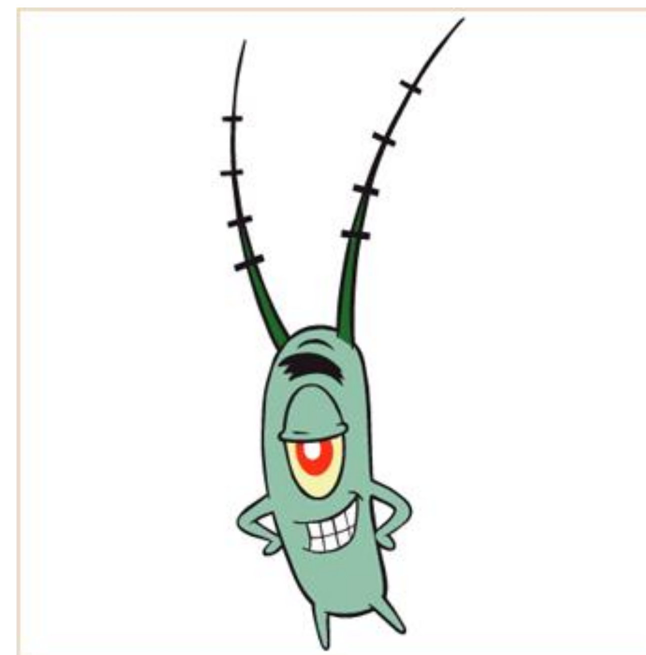
Thus g* prefers b* to b, and (b,g*) is not a rogue couple.

Opinion Poll

Who is better off in traditional dating?



The boys?



The girls?

My best girl

Forget about TMA for a minute...

How can we define “the optimal girl” for a boy b?

First attempt: b’s favorite girl.

This is unrealistic...in general b cannot hope to get his favorite girl in a stable world.

The optimal girl

Look at all possible stable pairings. Call a girl optimal for a boy if she is the highest ranked girl he is paired with in some stable pairing.

This is the best girl he can hope to get in a stable world.

Similarly, call the lowest ranked girl a boy can get in a stable pairing his pessimal girl.

Dating ups and downs

Call a pairing male-optimal if all males simultaneously are paired with their optimal female.

Call a pairing male-pessimal if all males simultaneously are paired with their pessimal female.

Call a pairing female-optimal if all females simultaneously are paired with their optimal male.

Call a pairing female-pessimal if all females simultaneously are paired with their pessimal male.

The TMA always produces a male-optimal female-pessimal pairing.

Male Optimal

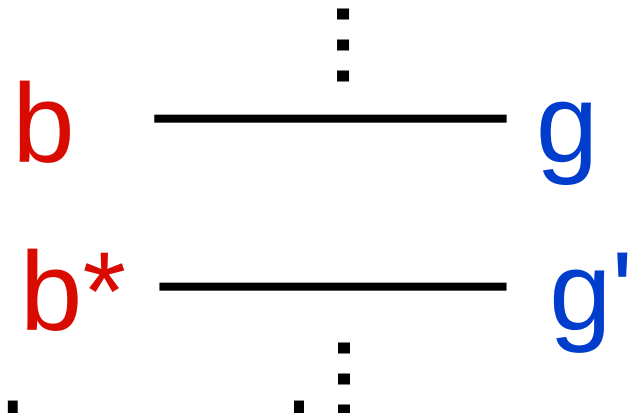
Proof by contradiction: Suppose not male optimal.
Let t be the earliest any boy is rejected by his optimal girl.

At time t optimal girl g rejects b for a preferable boy b*.
By definition of t, boy b* has not been rejected by his optimal girl.

Thus b* likes g at least as much as his optimal girl g*.

Male Optimal

By assumption that g is optimal for b, there is a stable pairing matching them together.



But g prefers b* to b, and
b* prefers g to g* (his optimal girl) and
prefers g* to g’ (this matching is stable, so g’ cannot be better than b’s optimal girl).

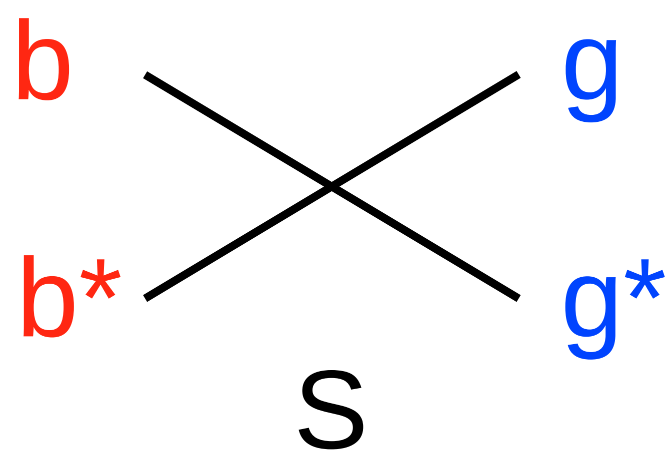
b* and g form a rogue couple! Contradiction.

Female-Pessimal

Male-optimal implies female pessimal

Proof by contradiction.

Suppose in a male-optimal pairing boy b is with girl g, yet there is some stable pairing S in which g is with less preferable b*.



But then b prefers g to g* (she is his optimal girl) and g prefers b to b*. Contradiction to stability of S.

Modeling

Assumptions:

- **Two** sided matching
- **Stability** is goal
- Each side **ranks** the other

Questions:

- Does there **always exist** a stable matching?
- Can we find it **efficiently**?
- Is it **unique**? If not what **kinds** of solutions exists.
- How much **utility** is sacrificed for stability?
- Is two-sided assumption **necessary**?

The Match

TMA is the algorithm used successful in the world.

? ? ?

No! By the National Residency Matching Program.

This service pairs graduating medical students with hospitals.

The Match

- Since 1952 residency matches have been made by a centralized service.
- Medical students rank hospitals.
- Hospitals rank graduating students
- They run the TMA (with some modifications).
- Until recently, pairings were hospital optimal; now resident optimal.

Question for thought

Couples can enter the match together now. They rank pairs of residency programs (usually for geographical reasons), and are matched as a pair instead of as individuals.

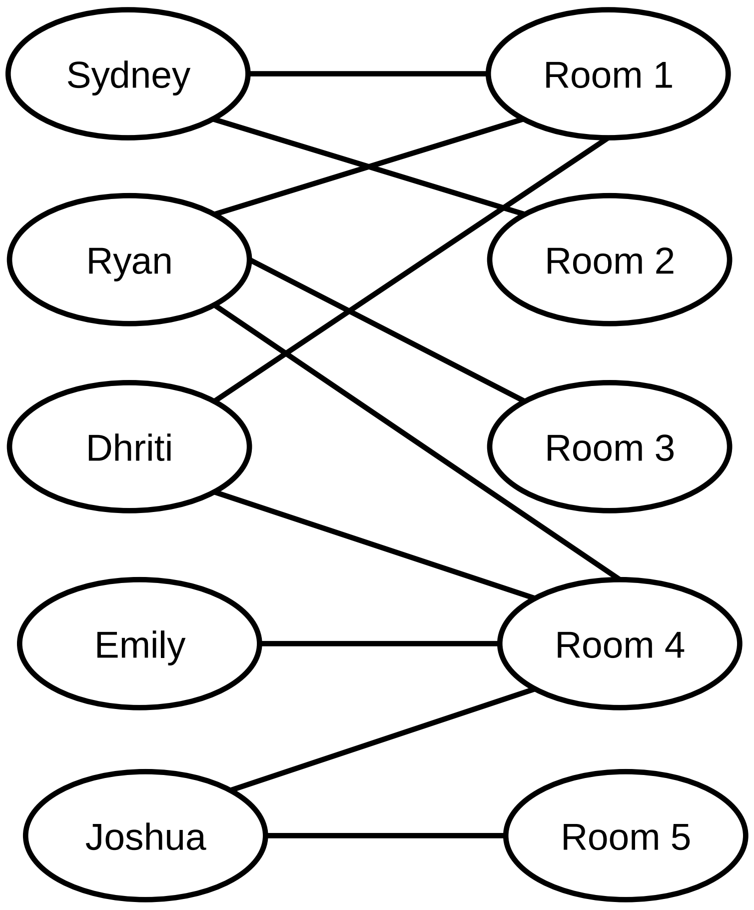
Show that in this case stable matchings need not exist.

History and Beyond

This algorithm was developed by Gale and Shapley back in 1962 in a paper “College Admissions and the stability of marriage.”

Bipartite Graphs

- Everyone says “Yeah” or “Nea” for each other person/room.



Bipartite Graphs

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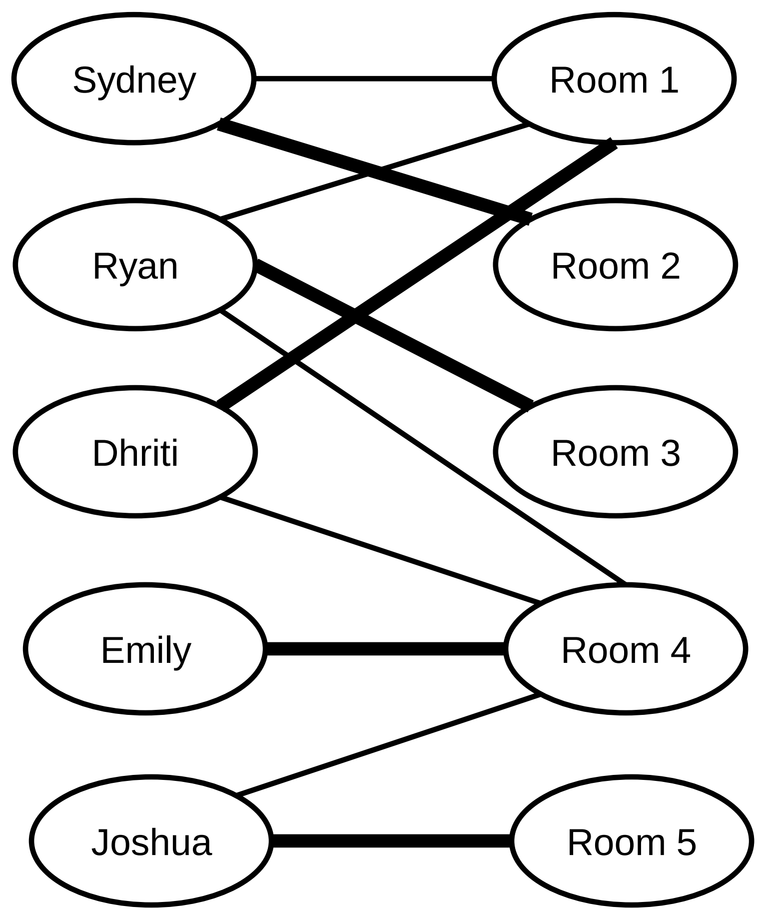
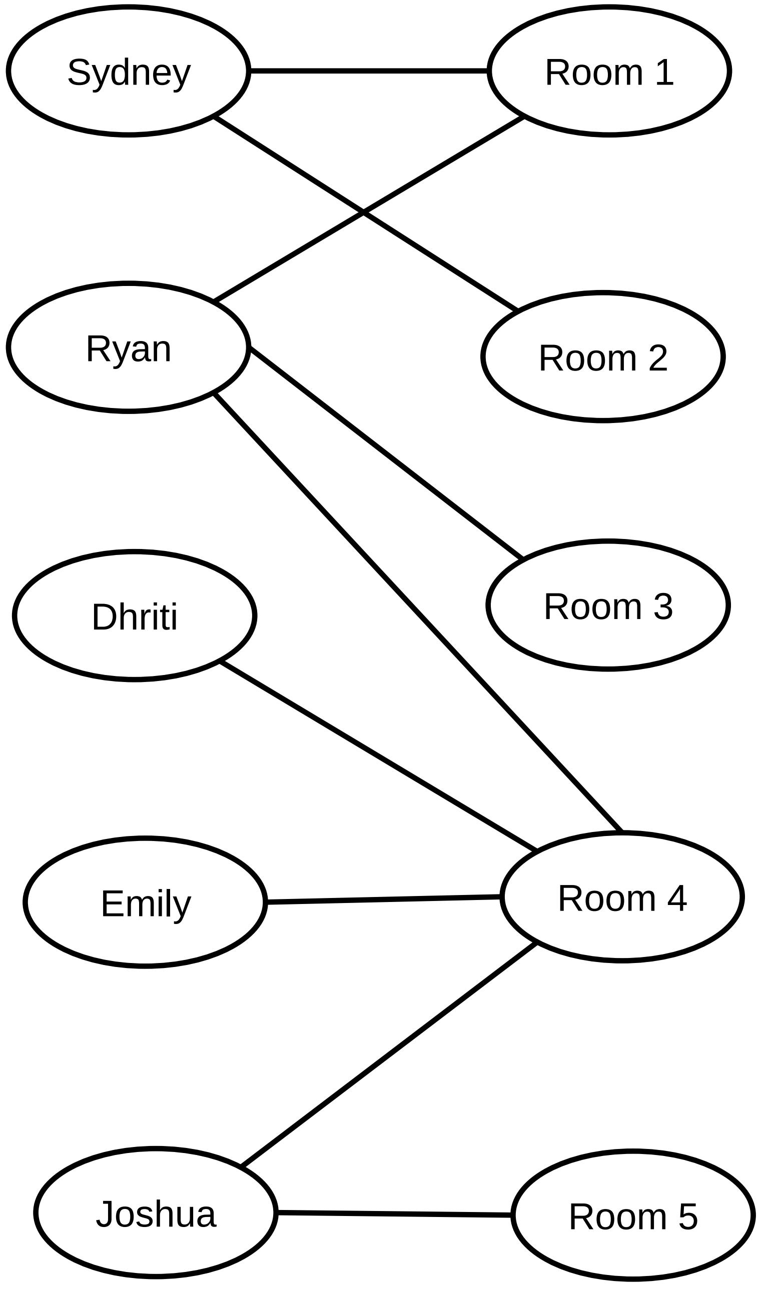
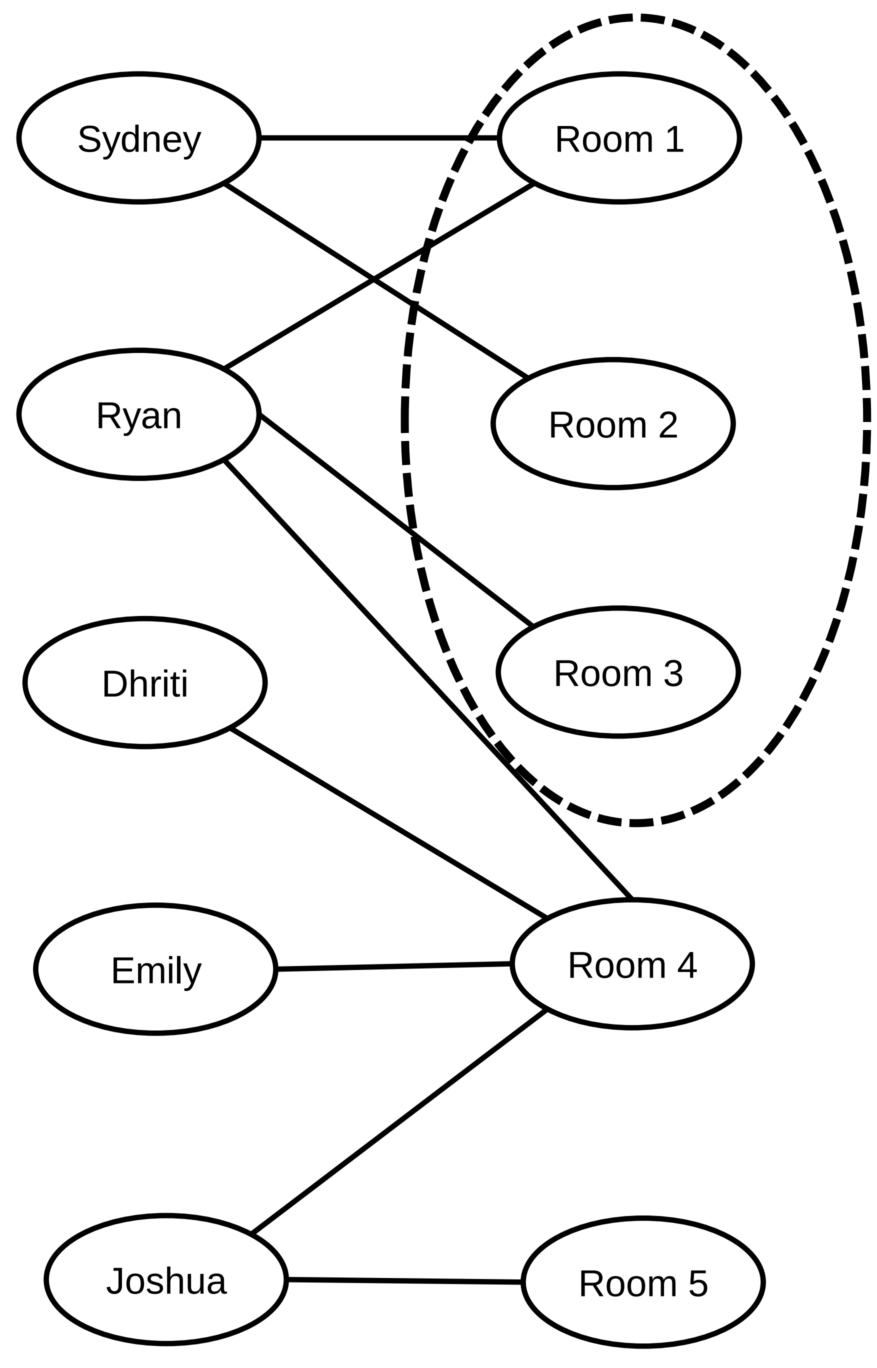


a perfect matching

Perfect Matching?



Constricted Set



Matching Theorem

- If a bipartite graph has no perfect matching, then it contains a constricted set.
- Konig 1931; Hall 1935
- Will prove later!

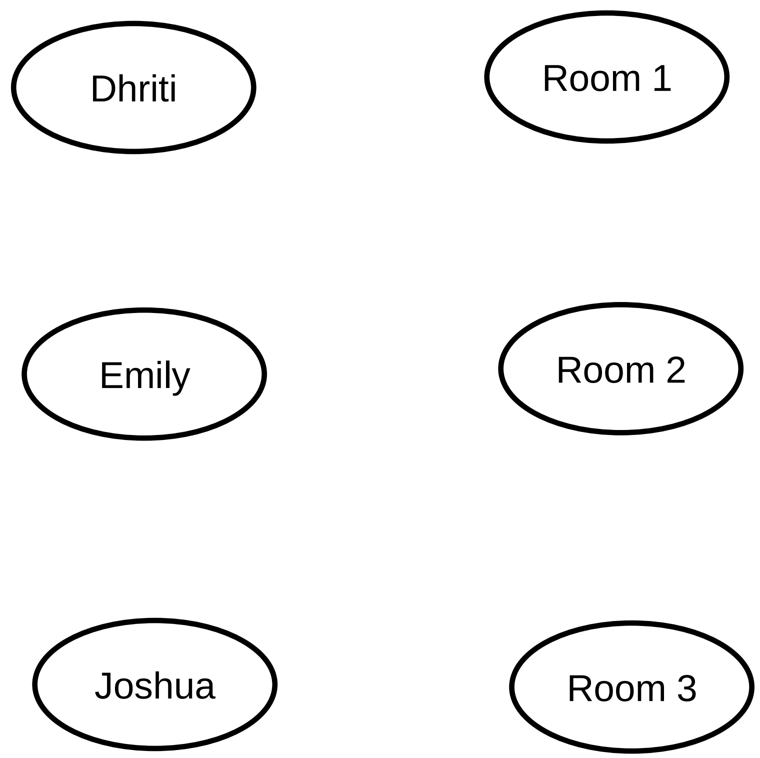
Optimal Assignments

Valuations

12, 2, 4

8, 7, 6

7, 5, 2



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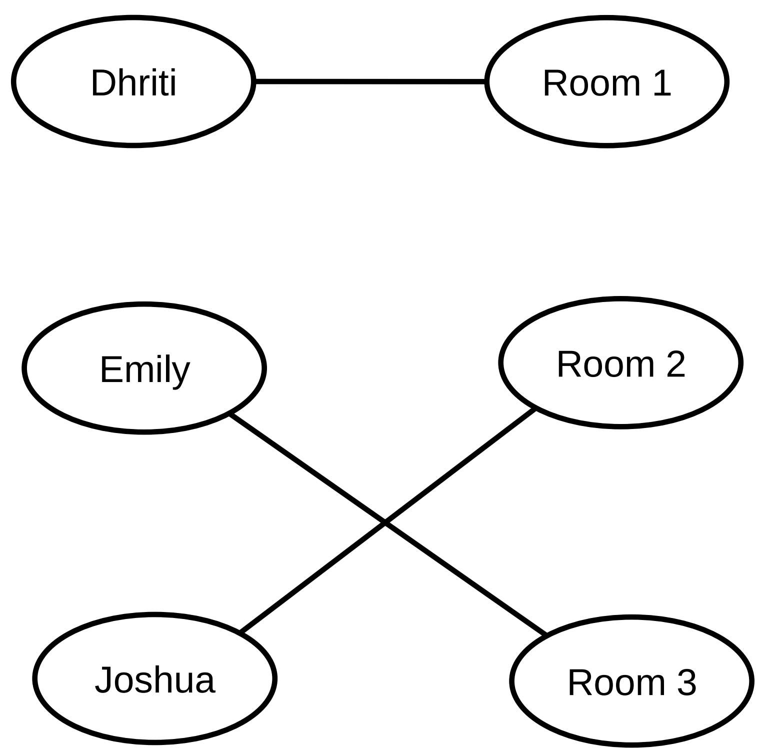
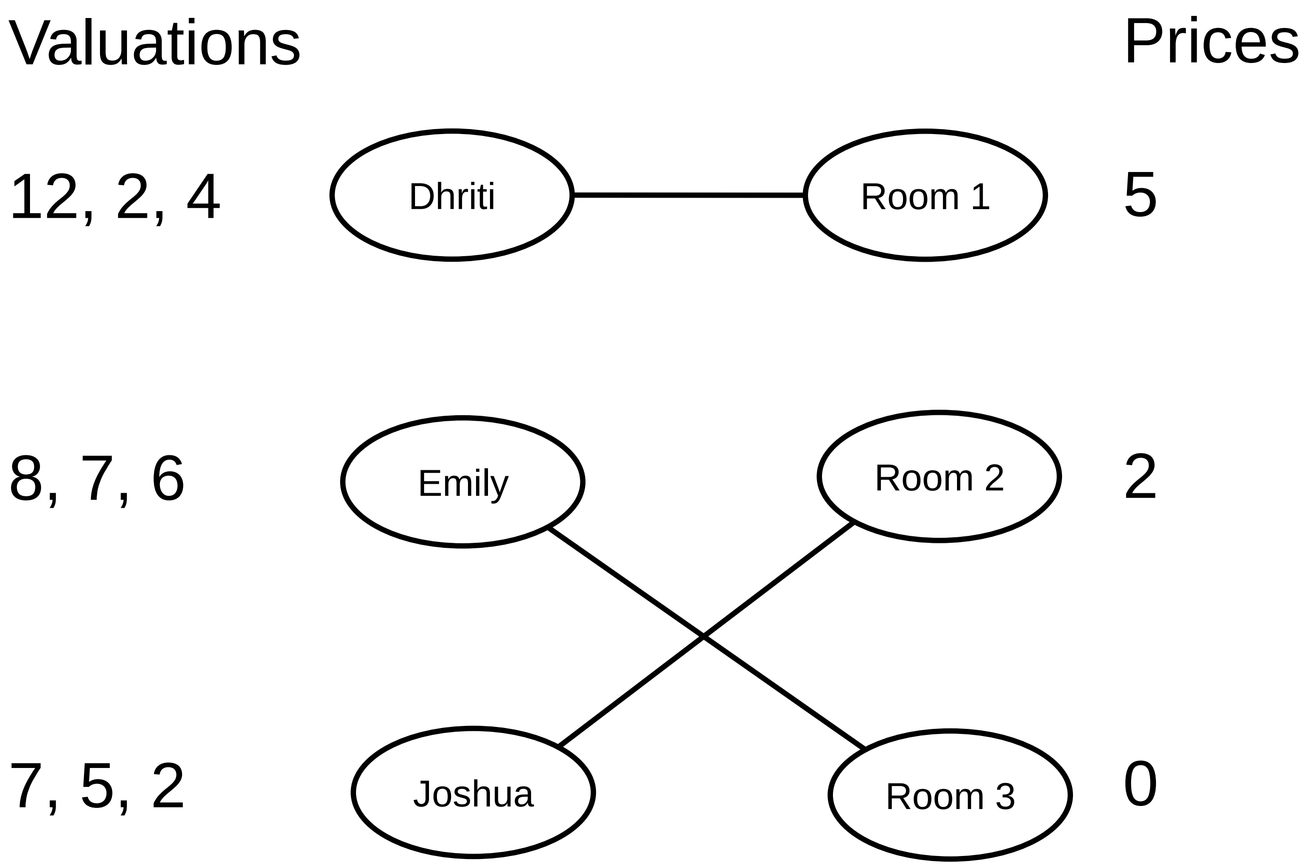


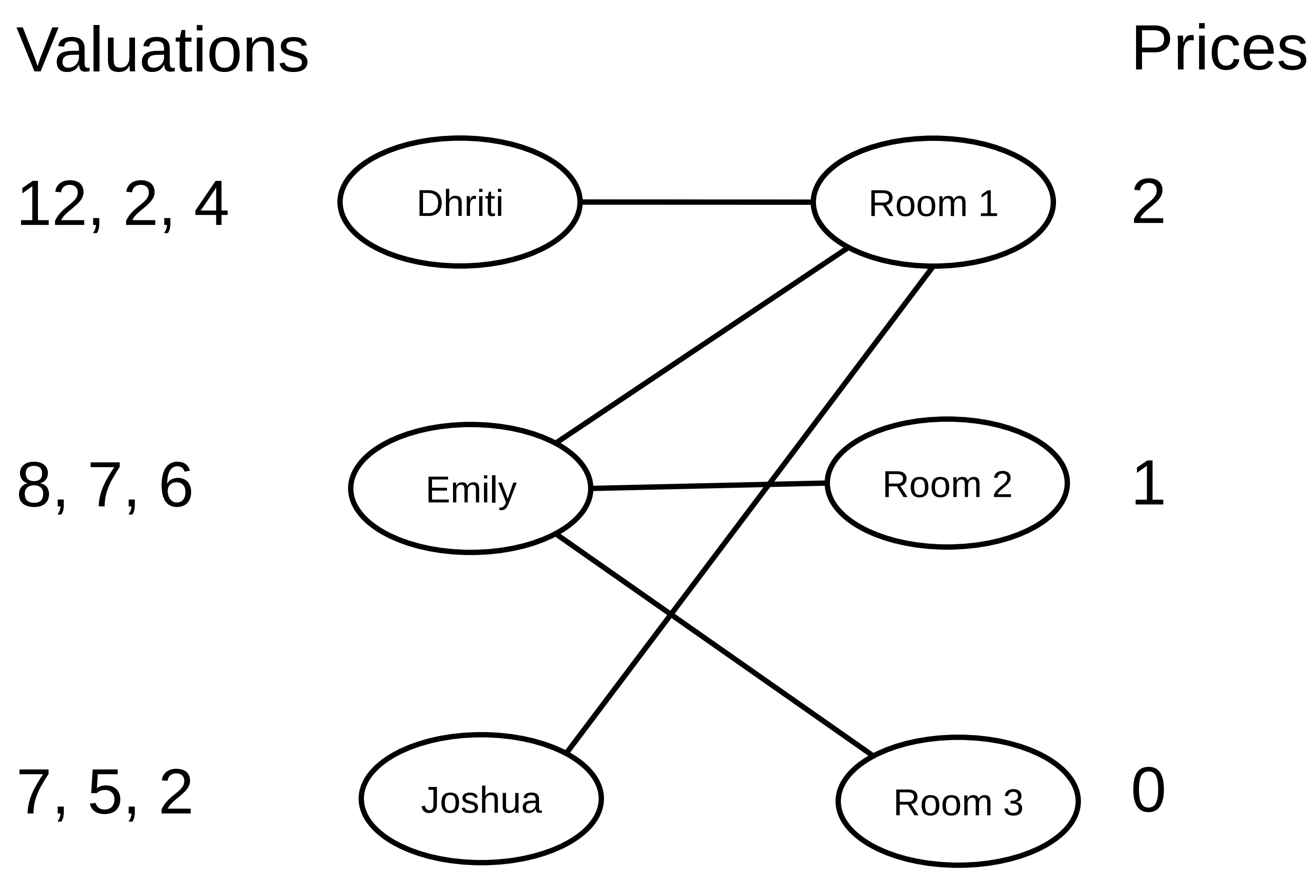
An Optimal Assignment

Prices Supporting Optimal Assignment

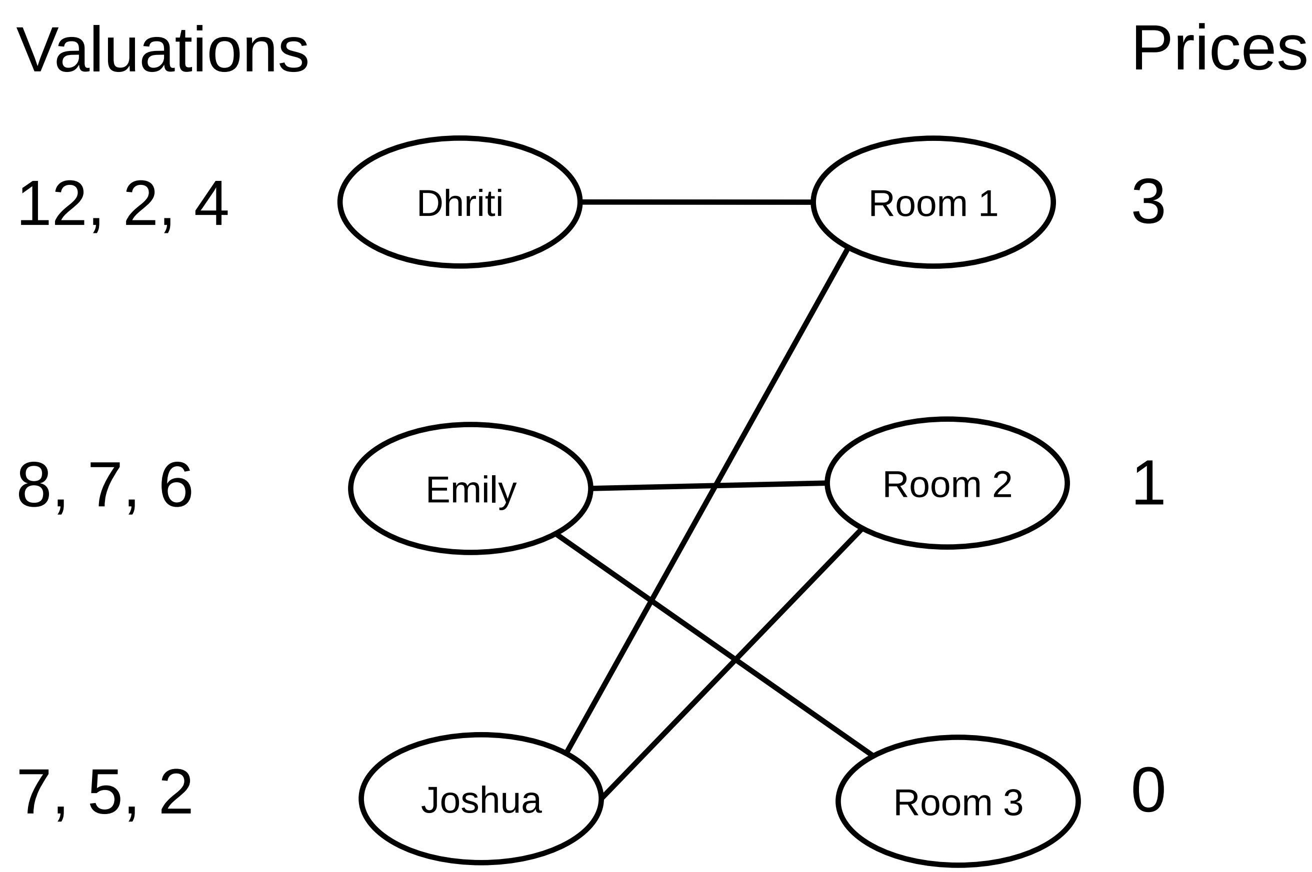


An Optimal Assignment
With supporting prices

Prices that do not clear market



Prices that clear



Requires tie breaking

Market Clearing Prices

- Do MCP always exist?
- What is the connection between MCP and Efficiency?
- Theorem: For any set of buyer valuations, there exists a set of market-clearing prices.
- Theorem: For any set of MCPs a perfect matching in the preferred seller graph has maximum total valuation of any assignment to buyers.

Proof of Optimality

- Fix prices.
- Sum of payoffs = Sum of valuations – Sum of prices
- But we are maximizing sum of payoffs!
- Sum of prices is fixed, so we are maximizing sum of valuations!

Proof of Optimality

- Let $b_{i,j}$ be buyer i's value for house j.
- Let c_i be cost for house j. Let $C = \sum_j c_j$
- Total value in market is then
- $C + \max_f \sum_i b_{i,f(i)} - c_{f(i)} =$
  $\mathrm{C} + \max_\sigma \sum_i b_{i,\sigma(i)} - c_{\sigma(i)} =$
  $$\max_\sigma \sum_i b_{i,\sigma(i)}$$

Hungarian Algorithm

- Start All prices at 0.
- Each Round:
  - 1) There is a current set of prices with smallest one equal to 0
  - 2) Construct preferred Seller Graph. PM?
  - 3) If PM, we are done, if not let S be constricted set of buyers with neighbors N(S)
  - 4) Each seller in N(S) raises prices by 1
  - 5) Uniformly reduce prices if necessary, and go to 1)

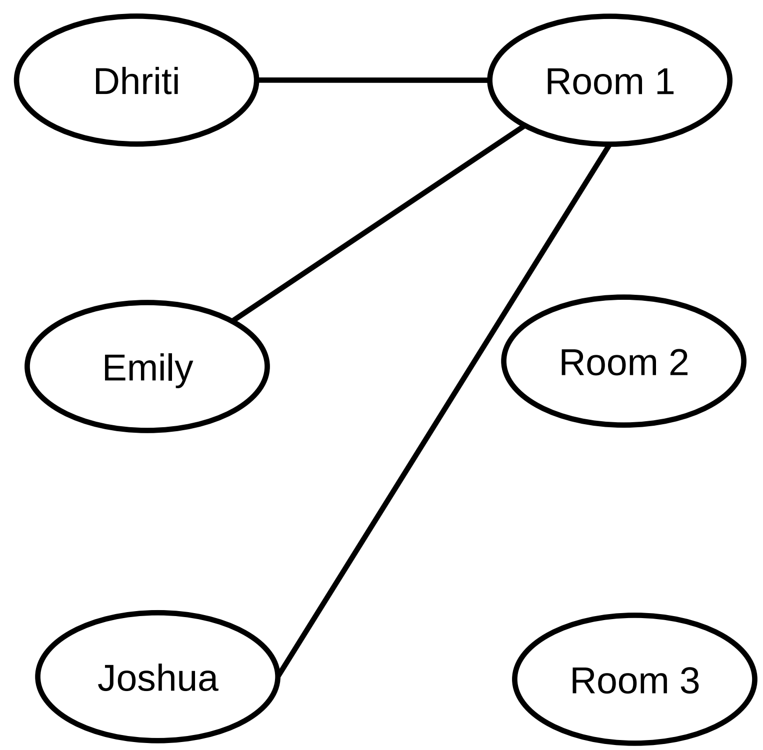
Run the Algorithm

Valuations

12, 2, 4

8, 7, 6

7, 5, 2



Prices

0

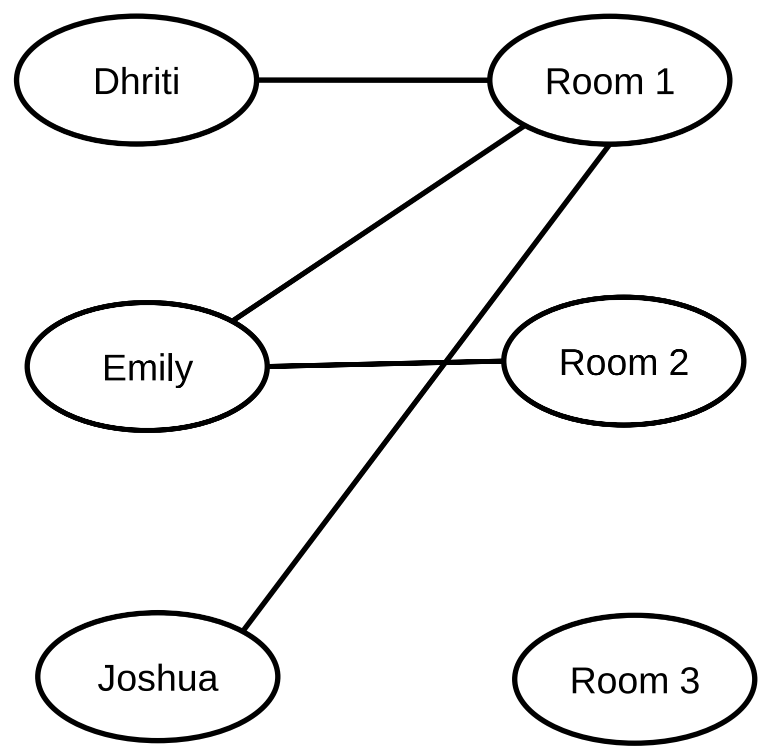
0

0

Round 0

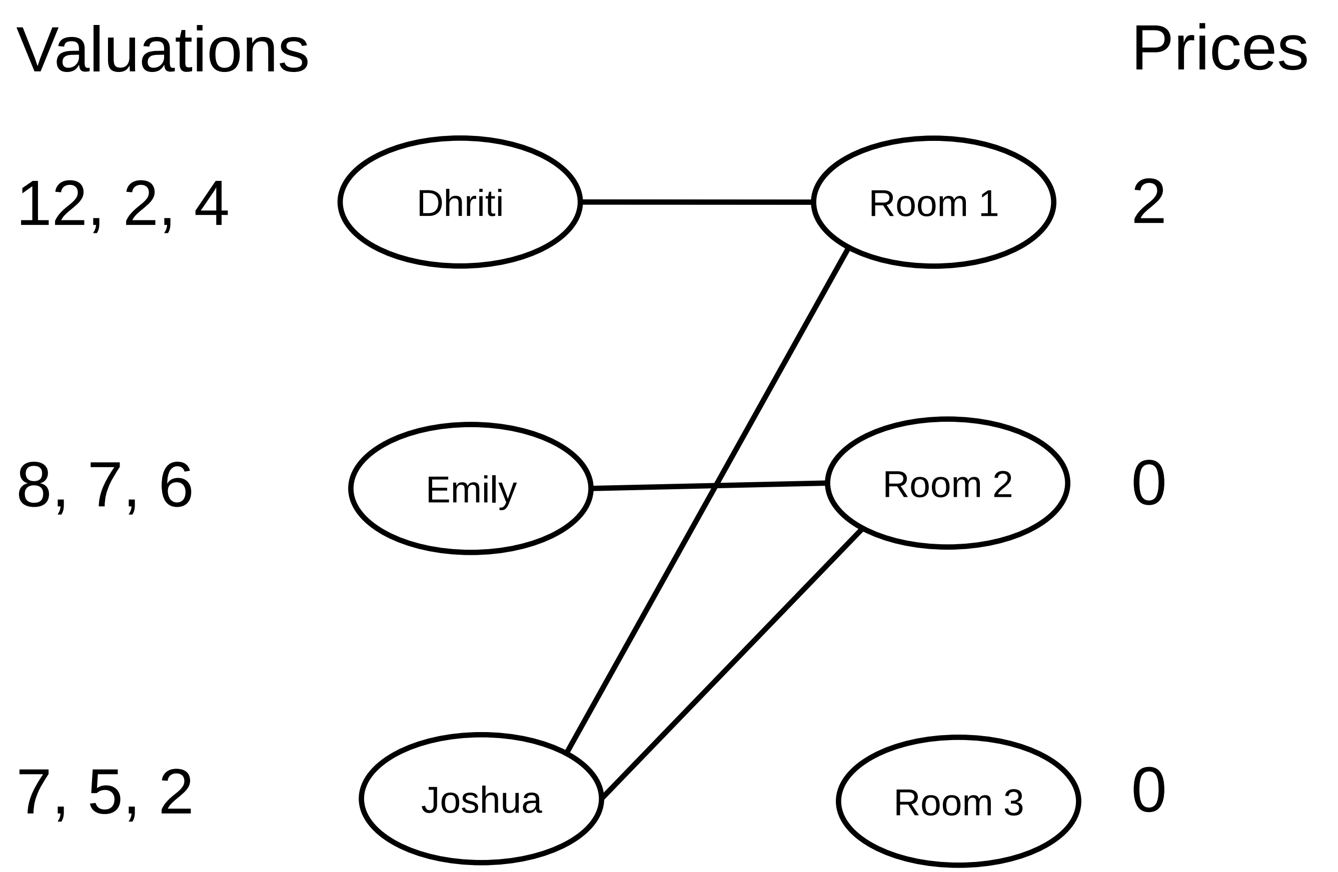
Run the Algorithm

Valuations		Prices
12, 2, 4		1
8, 7, 6		0
7, 5, 2		0



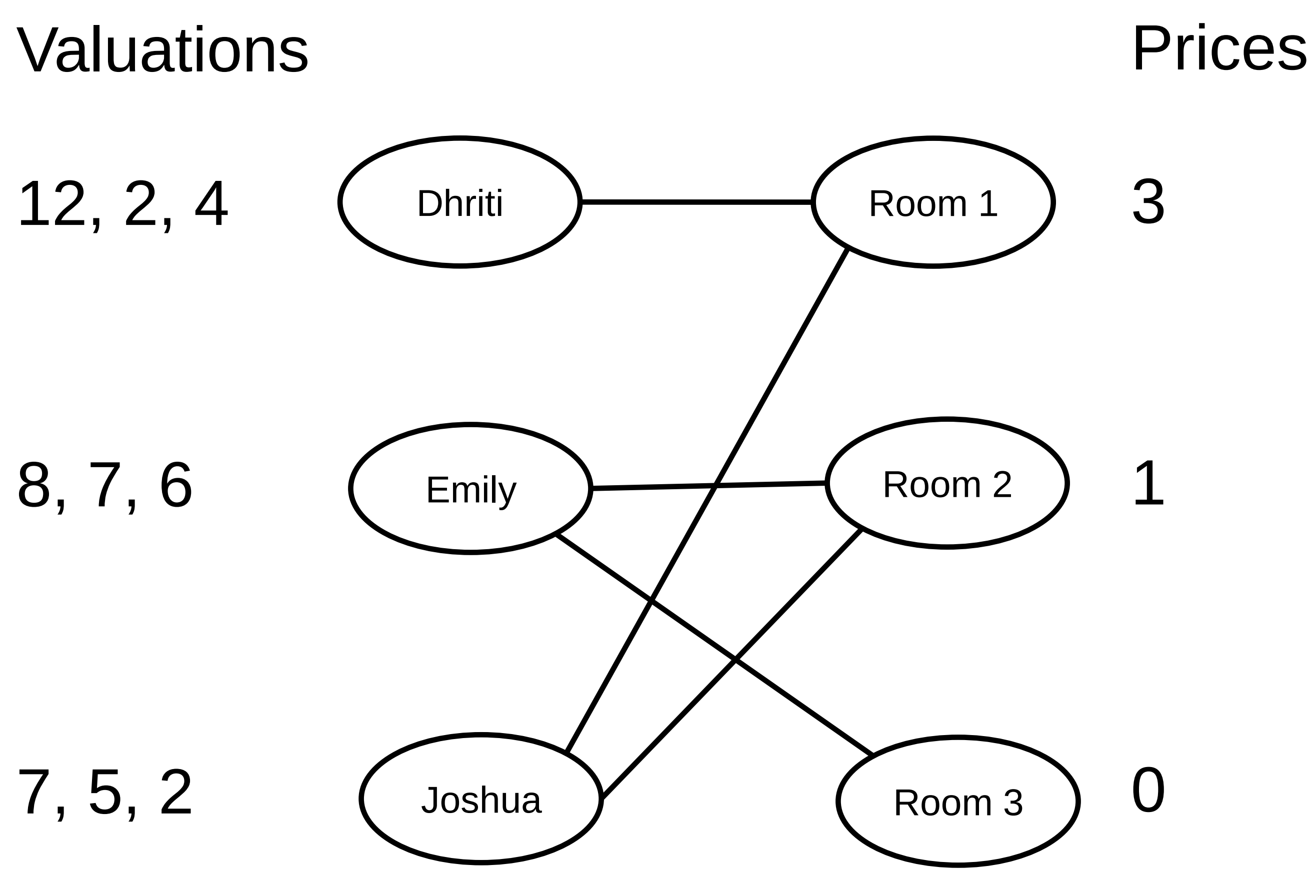
Round 1

Run the Algorithm



Round 2

Run the Algorithm



Round 3

Proof of termination

- Look at the sum of current payoffs in the preferred seller graph (both buyers and sellers)
- This is always positive, and decrease by at least one in each round.

Back to Matchings

- This Provides an algorithm for finding maximum weighted matchings!
- Prices provide a succinct proof of optimality
- Is this mechanism truthful?