Computability

Limits of Computation

Attribution

These slides were prepared for the New Jersey Governor's School course "The Math Behind the Machine" taught in the summer of 2011 by Grant Schoenebeck

Large parts of these slides were copied or modified from the previous years' courses given by Troy Lee in 2010 and Ryan and Virginia Williams in 2009.

Turing's Legacy: The Limits Of Computation.

David Hilbert (1862-1943)

Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?

In mathematics there is no ignorabimus.

The HELLO WORLD assignment

Suppose your teacher tells you:

Write a JAVA program to output the word "HELLO WORLD" on the screen and halt.

Space and time are not an issue.
The program is for an ideal computer.

PASS for any working HELLO program, no partial credit.

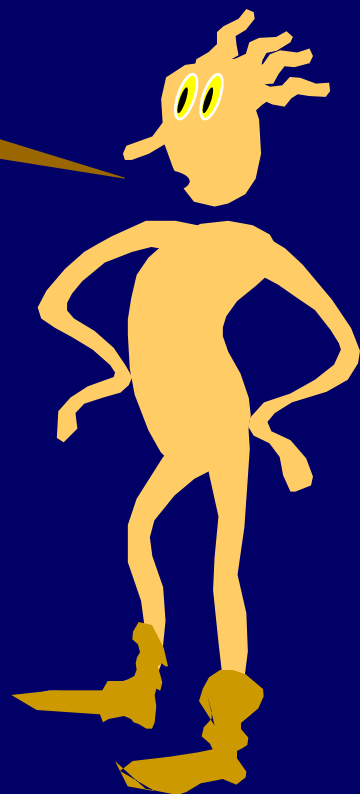
Teacher's Grading Program

The grading program G must be able to take any Java program P and grade it.

$$G(P) = \begin{cases} \text{Pass, if P prints "HELLO WORLD"} \\ \text{Fail, otherwise.} \end{cases}$$

How exactly might such a script work?

What kind of program could a student who hated his/her teacher hand in?



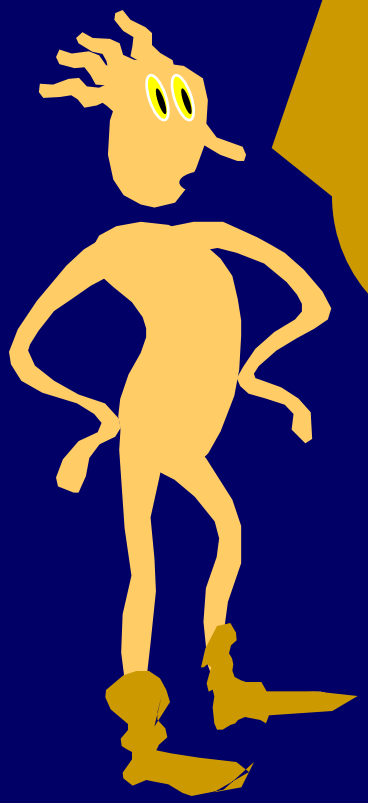
Nasty Program

```
n:=2;
While  (the number 2n can be written as the sum of two primes)
        n++;
Print "HELLO WORLD";
```

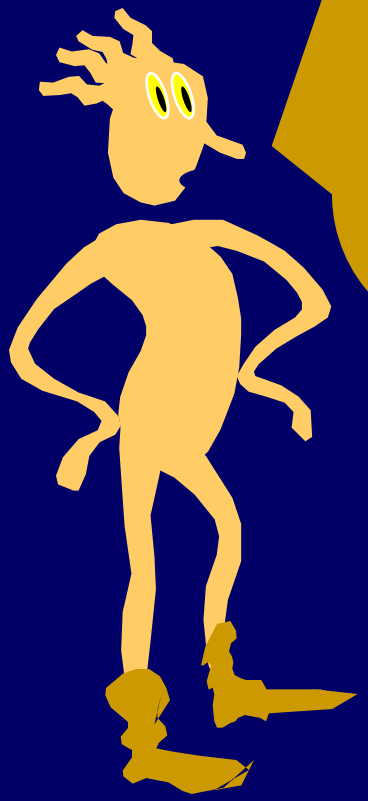
The nasty program is a PASS if and only if the Goldbach conjecture is false.

Despite the simplicity of the HELLO WORLD assignment, there is no program to correctly grade it!

This can be proved.



The theory of what can and can't be computed by an ideal computer is called Computability Theory or Recursion Theory.



The Ideal Memory Model

Σ = finite alphabet of symbols

Each memory location holds one element of Σ

"Abstract" Version: One memory location for each natural number 0, 1, 2, …

"Practical" Version: Any time you start to run out of memory, the computer contacts the factory. A maintenance person is flown by helicopter and attaches 100 Terabytes of RAM to the computer.

Computable Functions

Fix any precise programming language, i.e., Java.

A program is any finite string of symbols from Σ that a Java interpreter will run (won't give a syntax error)

Recall Σ^* is the set of all strings of symbols.

A function $f : \Sigma^* \rightarrow \Sigma^*$ is computable if there is a program P that computes f, when P is executed on a computer with ideal memory.

That is, for all strings x in Σ^*, P(x) = f(x).

The set of all programs is a countable set!

Fix any precise programming language, i.e., Java.

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That is, for all strings x in Σ^*, P(x) = f(x).

The set of all computable functions is also a countable set!

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Recall Σ^* is the set of all strings of symbols.

A function $f : \Sigma^* \rightarrow \Sigma^*$ is computable if there is a program P that computes f, when P is executed on a computer with ideal memory.

That is, for all strings x in Σ^*, P(x) = f(x).

There are "countably many" Java programs. Hence, there are only "countably many" computable functions.

Are there countably many functions from Σ^* to Σ^* ?

Power Sets

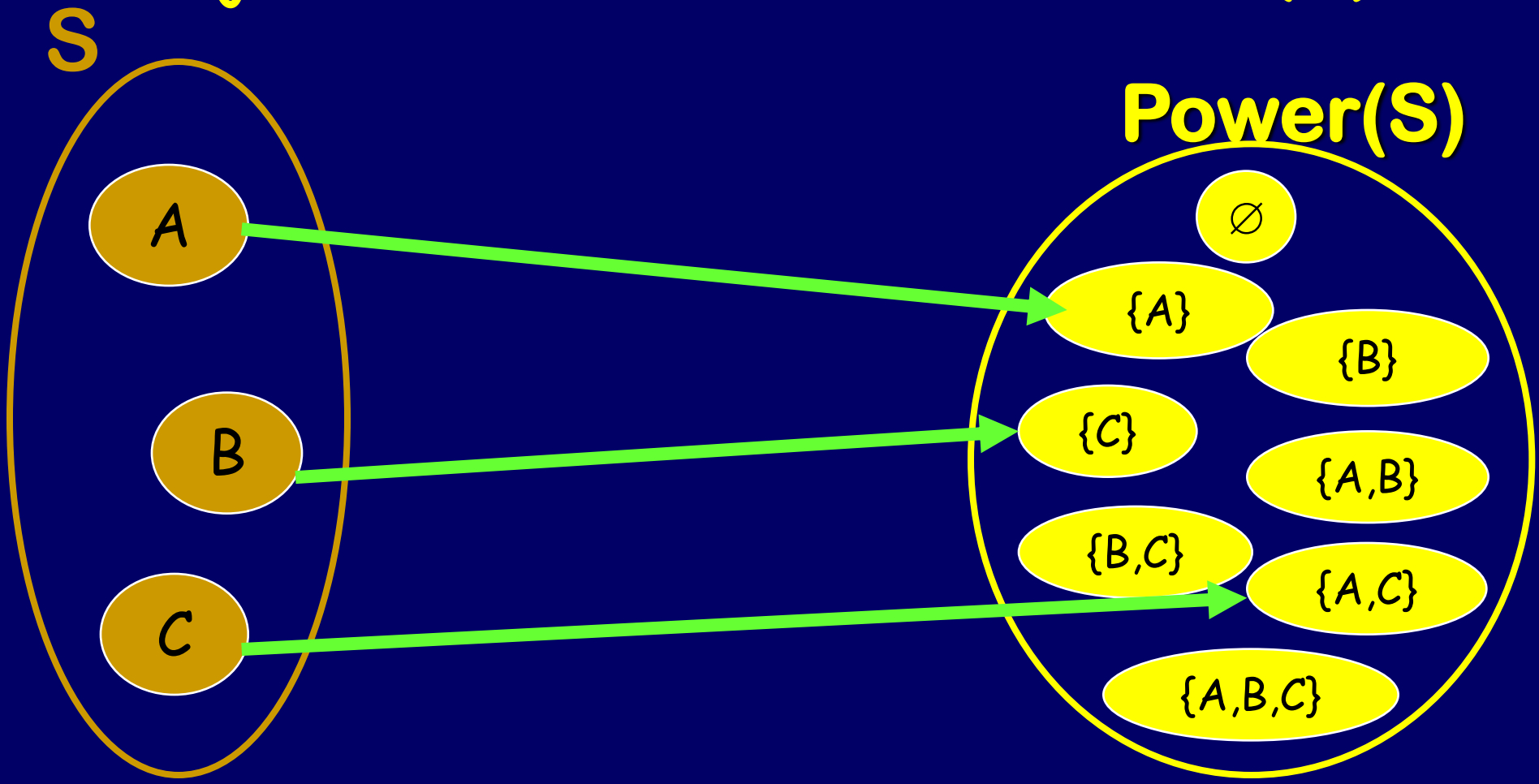
Let S be a set.

The power set of S is the set of all subsets of S.

We write the power set as Power(S).

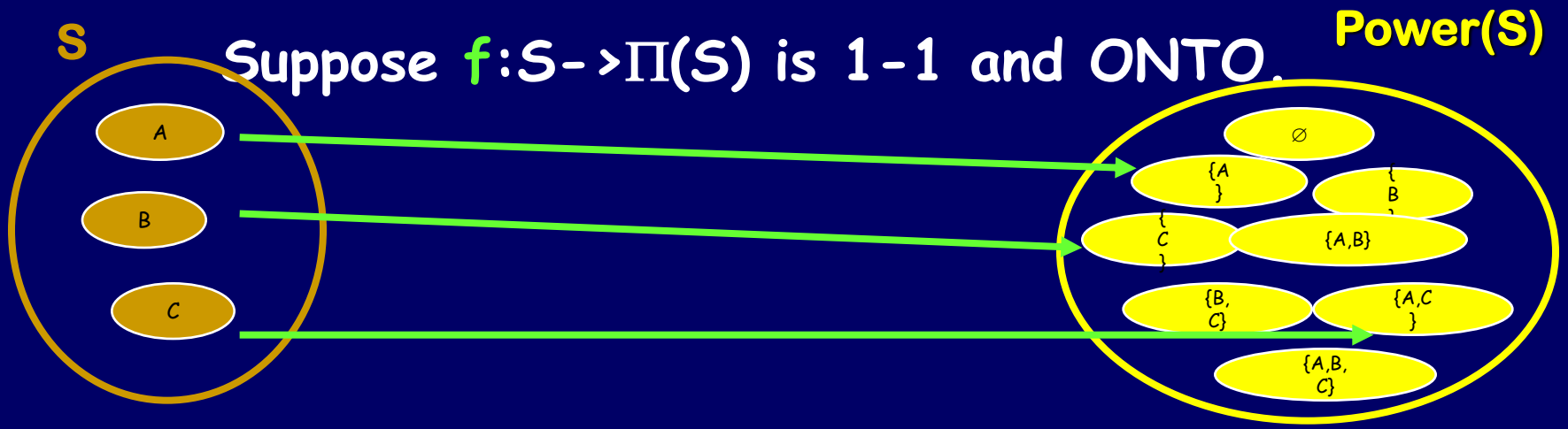
Proposition: If S is finite, then Power(S) has cardinality $2^{|S|}$

Theorem: For every S, there is no bijection between S and Power(S)



Suppose f : S->Π(S) is a bijection.

Theorem: For every S, there is no bijection between S and Power(S)



Let WEIRD = { x | x ∈ S, x ∉ f(x) }

There's some y in S such that f(y)=WEIRD

Is y in WEIRD? YES or NO?

if y in WEIRD, then y ∈ S and y ∉ f(y) = WEIRD

So y is not in WEIRD...

Contradiction

y ∈ S and y ∉ WEIRD = f(y)... So y is in WEIRD...

Theorem: There are uncountably many functions!

There is a bijection between
- The set of all subsets of Σ^*
(the powerset of Σ^*)
- The set of all functions f: Σ^* -> {0,1}

Take a subset S of Σ^*, we map it to the function f where:

f(x) = 1 x in S

f(x) = 0 x not in S

Uncountably many functions.

There is a bijection between
- The set of all subsets of Σ^*
(the powerset of Σ^*)
- The set of all functions f: Σ^* - > {0,1}

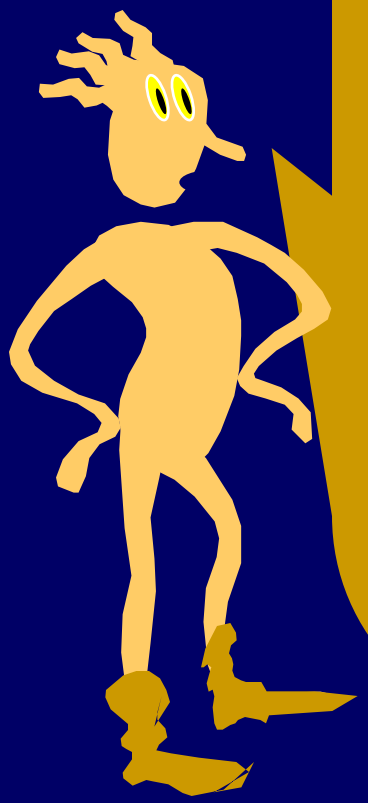
So the set of all f: Σ^* -> {0,1} has the same size as the powerset of Σ^*

But Σ^* is countable, so the powerset of Σ^* is uncountable!

(No bijection between Σ^* and Power(Σ^*)!)

So there are functions from Σ^* to {0,1} that are not computable.

Can we describe an incomputable one?
Can we describe an interesting, incomputable function?



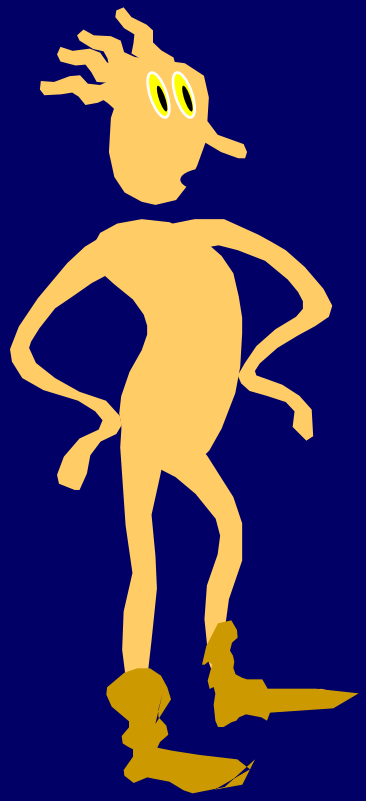
Notation And Conventions

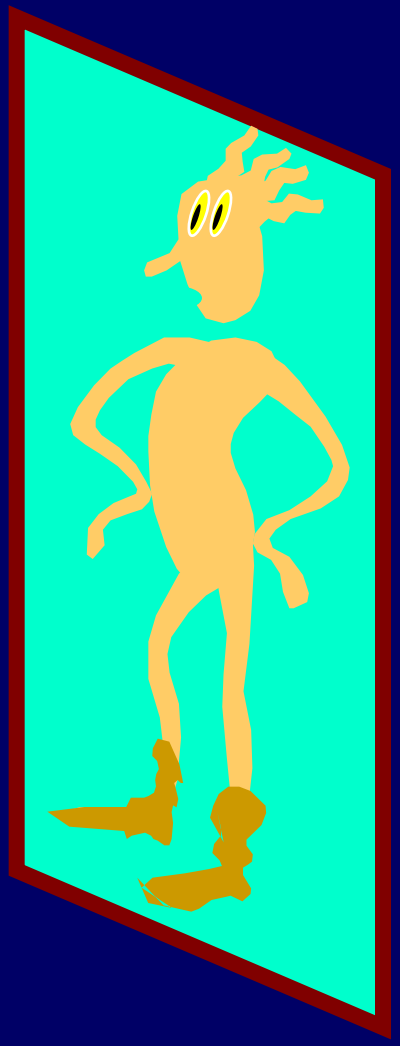
- Fix any programming language
- When we refer to “program P” we mean the text of the source code for P
- P(x) is the final output of program P on input x, assuming that P eventually halts

P(P)

It follows from our conventions that P(P) is the output obtained when we run P on the text of its own source code.

P(P) … So that's what I look like





The Famous Halting Set: K

K is the set of all programs P such that P(P) halts.

K = { Program P | P(P) halts}

The Halting Problem

Is there a program HALT such that:

HALT(P) = yes, if P(P) halts
HALT(P) = no, if P(P) does not halt

The Halting Problem
$K = \{P \mid P(P) \text{ halts } \}$

Is there a program HALT such that:

HALT(P) = yes, if $P \in K$
HALT(P) = no, if $P \notin K$

HALTS decides whether or not any given program is in K.

THEOREM: There is no program that can solve the halting problem! (Alan Turing 1937)

Suppose a program HALT, solving the halting problem, existed:

HALT(P) = yes, if P(P) halts
HALT(P) = no, if P(P) does not halt

We will call HALT as a subroutine in a new program called WEIRD.

The Program WEIRD(P):
If HALT(P) then go into an infinite loop.
Else stop.
<Put text of subroutine HALT here>

Does WEIRD(WEIRD) halt or not?

YES implies HALT(WEIRD) = yes
but then, WEIRD(WEIRD) will infinite loop

NO implies HALT(WEIRD) = no
but then, WEIRD(WEIRD) halts

The Program WEIRD(P):
If HALT(P) then go into an infinite loop.
Else stop.
<Put text of subroutine HALT here>

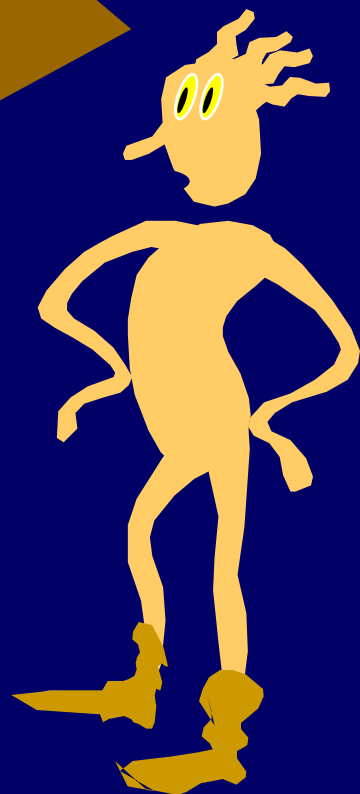
Does WEIRD(WEIRD) halt or not?

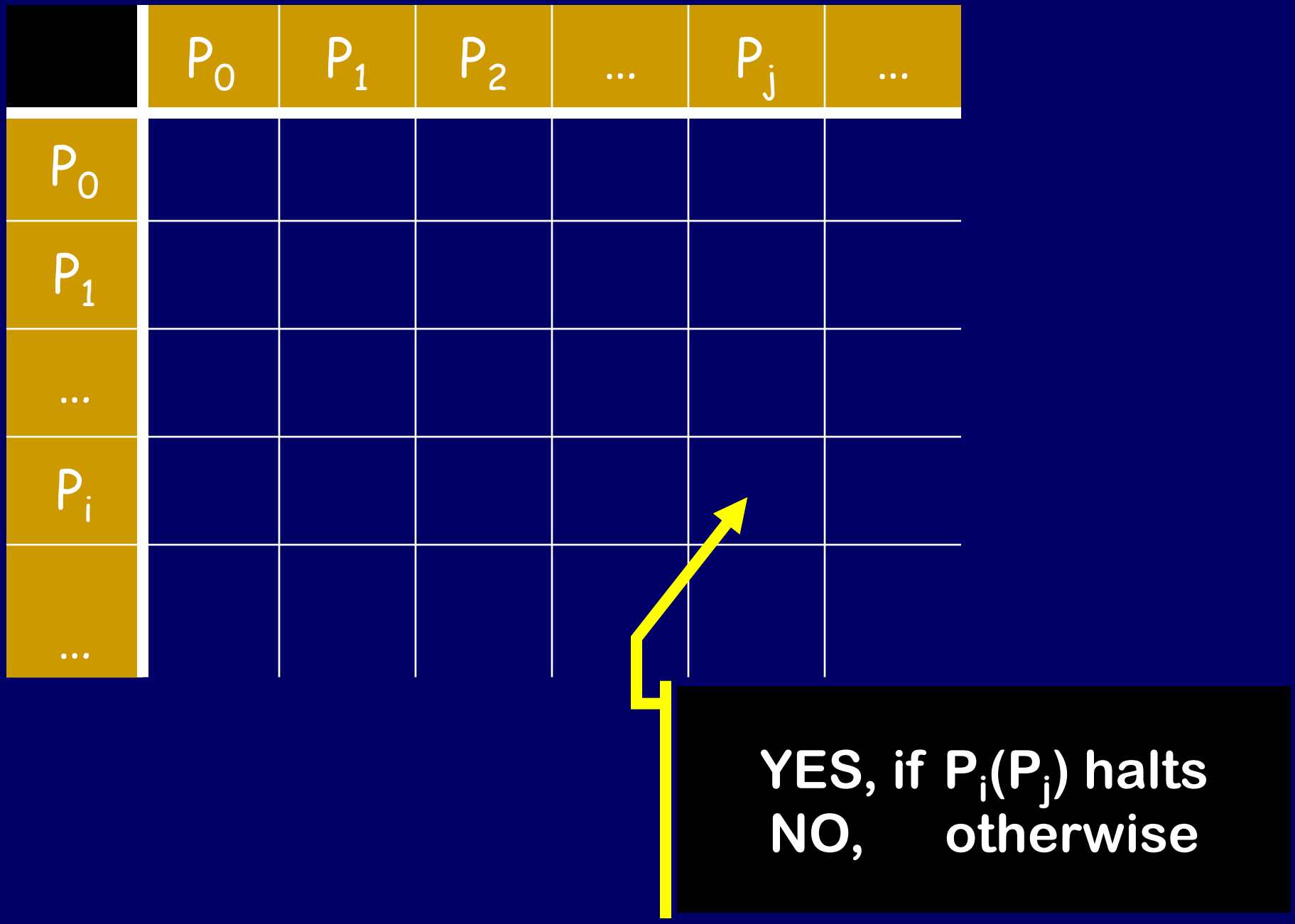
YES implies HALT(WEIRD) = yes
but then, WEIRD(WEIRD) will infinite loop

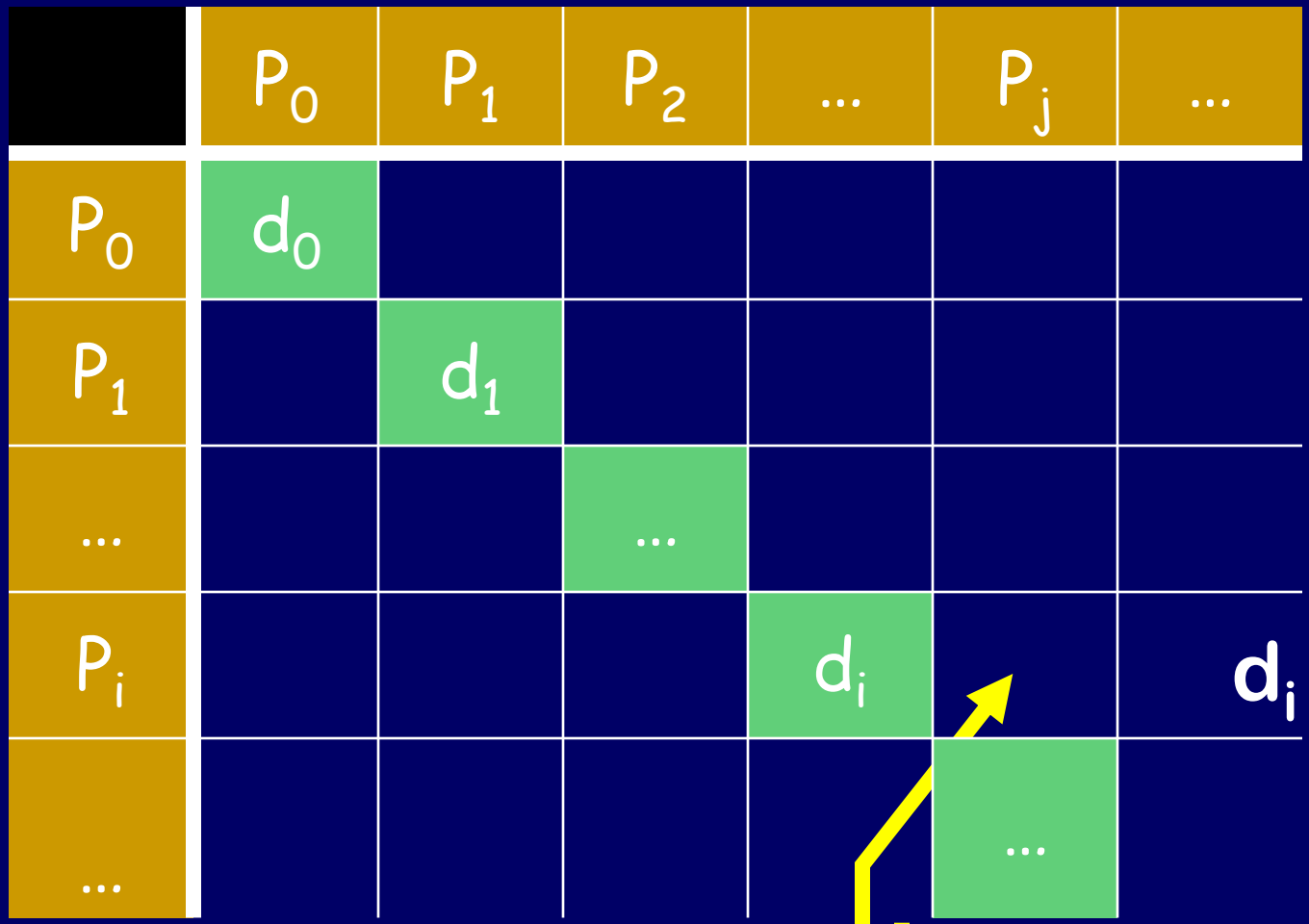
CONTRADICTION

NO implies HALT(WEIRD) = no
but then, WEIRD(WEIRD) halts

Turing's argument is just like the DIAGONALIZATION argument from the theory of infinities.

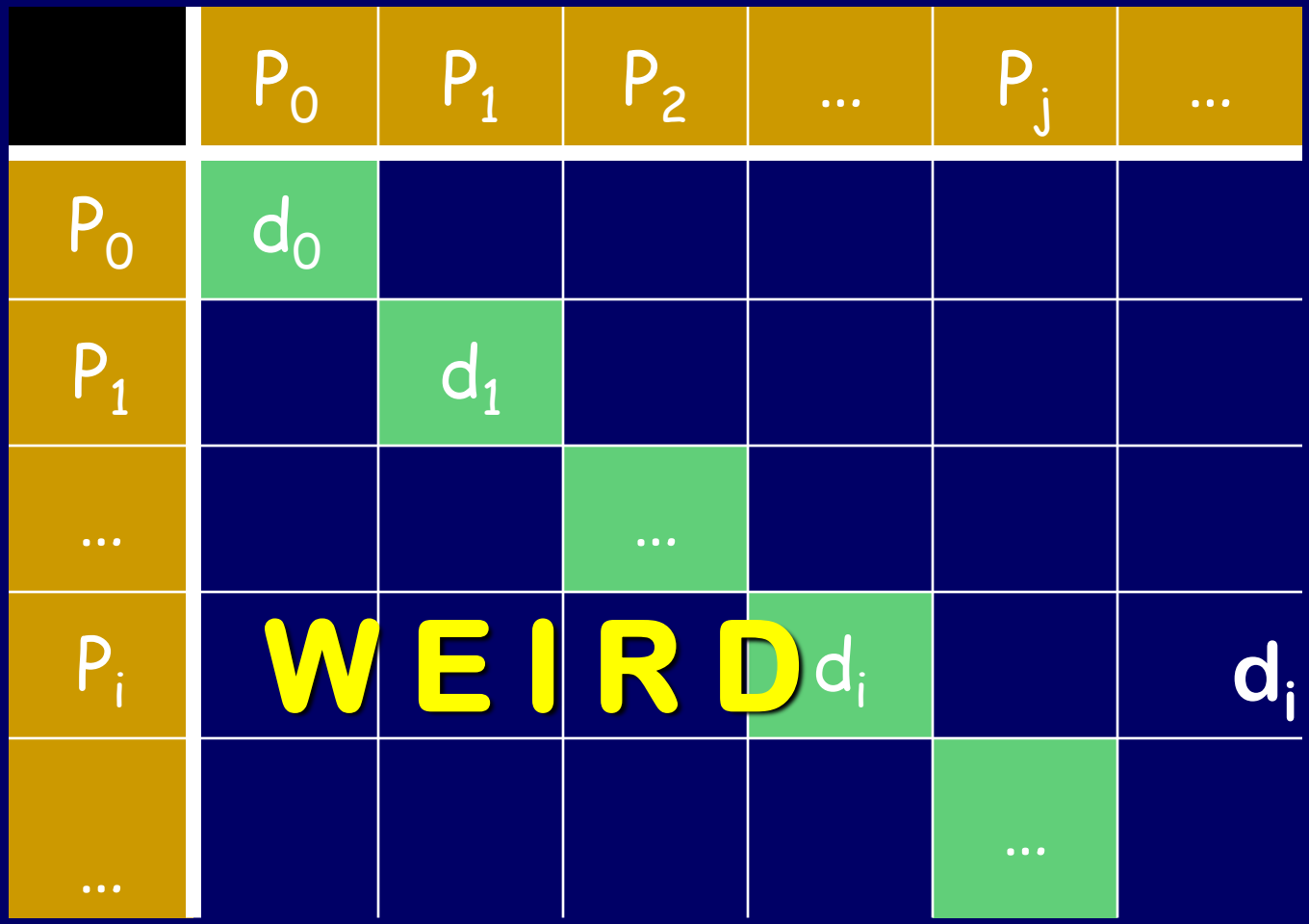



P_0 P_1 P_2 ... P_j ...
P_0
P_1
...
P_i
...
YES, if $P_i(P_j)$ halts
NO, otherwise



$d_i = HALT(P_i)$

YES, if $P_i(P_j)$ halts
NO, otherwise



WEIRD(P_i) halts iff d_i = NO
The WEIRD row contains the opposite of the diagonal…

Alan Turing (1912-1954)

Is there a real number that can be described, but not computed?

Consider the real number between 0 and 1, which has a 1 in the i^{th} decimal place if P_i **is in** K, and 0 otherwise

Computability Theory: Vocabulary Lesson

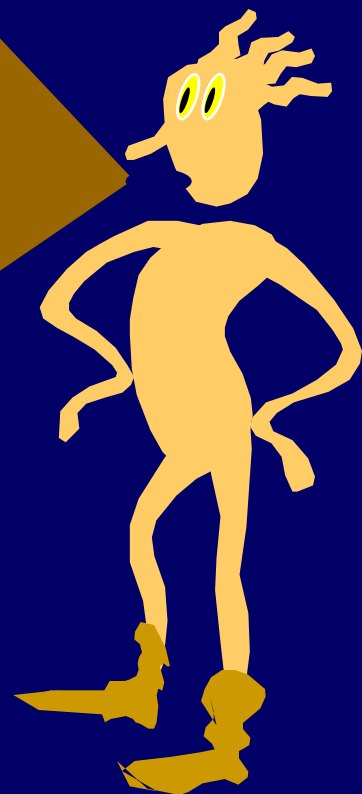
We call a set $S \subseteq \Sigma^*$ <u>decidable</u>
if there is a program P such that:

$P(x) = yes$, if $x \in S$

$P(x) = no$, if $x \notin S$

We already know: K is **undecidable**

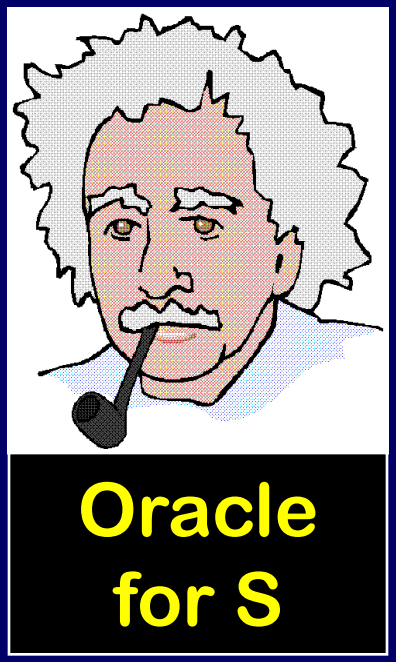
Now that we have established that the Halting Set K is undecidable, we can use it as a starting point for more “natural” undecidability results.



Oracle For Set S

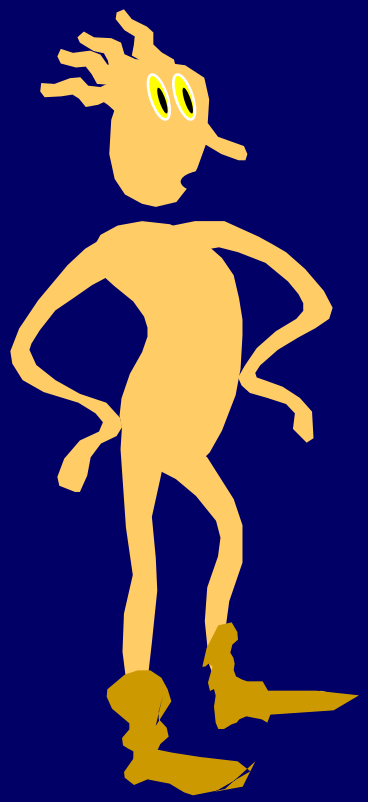
Is $x \in S$?

YES/NO



Example Oracle

S = Odd Naturals



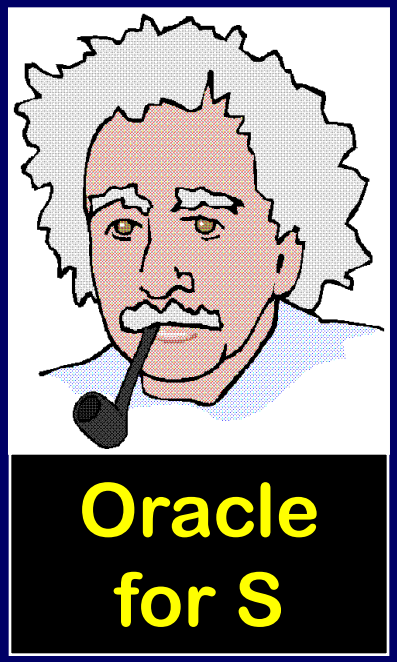
4

?

No

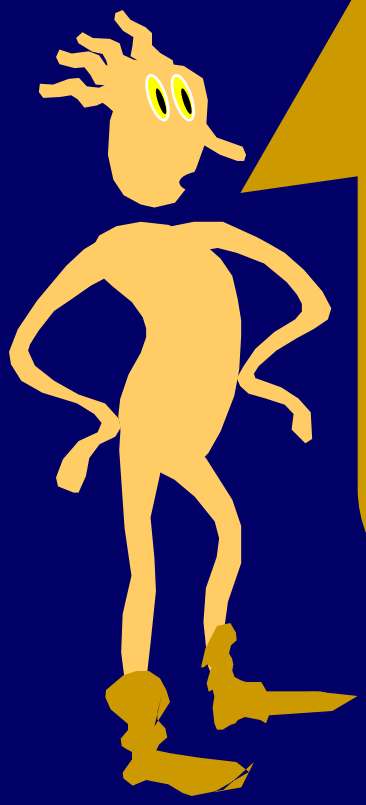
81?

Yes



Oracle for S

L = the set of programs that take no input and halt



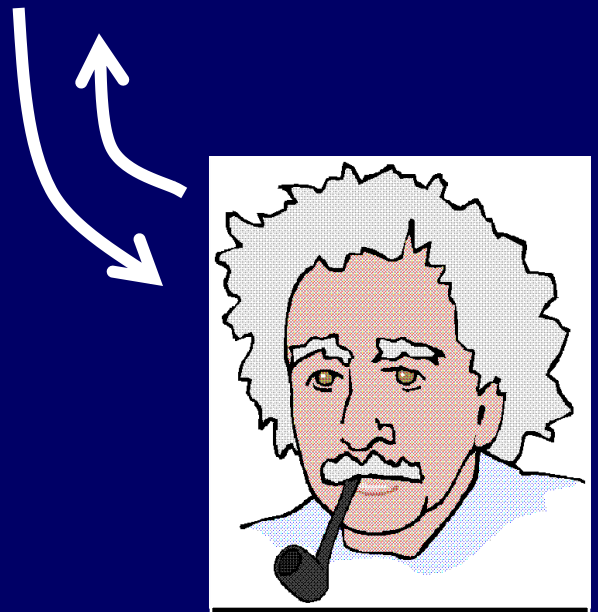
Hey, I ordered an oracle for the famous halting set K, but when I opened the package it was an oracle for the different set L.

GIVEN: Oracle for L

L = the set of programs that take no input and halt

P ; Q ≡ simulates P using P as input

Does P(P) halt?



Does the program Q halt?

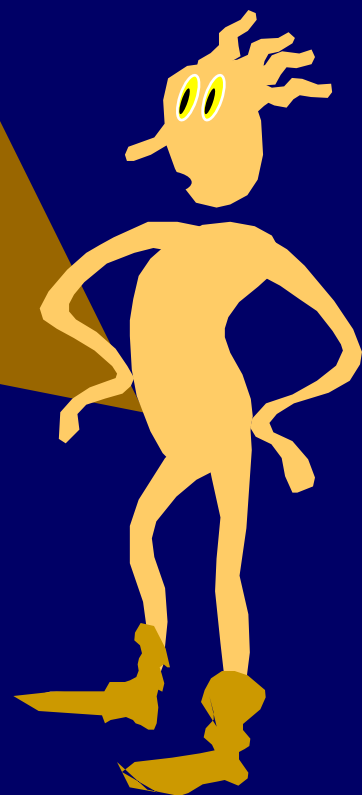
BUILD: Oracle for K

GIVEN: Oracle for L

Thus, if L were decidable then K would be as well.

(If there were a program for L, there'd be one for K, too!)

We already know K is not decidable. Therefore L is also not decidable!



HELLO = the set of programs that print HELLO and hal

Does P halt?

Let P' be P with all print statements removed.

BUILD:
Oracle
for L

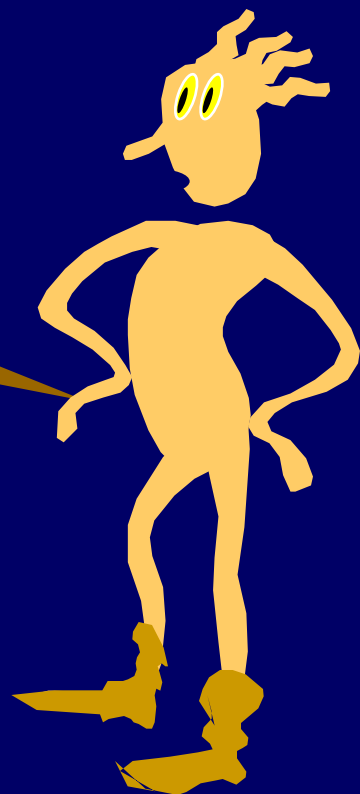
Does
[P'; Print HELLO]
ever print HELLO?

GIVEN:
HELLO
Oracle

If there were a program for HELLO, then there'd be a program for L.

But L is not decidable.

So HELLO is not decidable.



EQUAL = All <P,Q> such that P and Q have identical outputs on all inputs

Does P equal HELLO ?

Let H = [Print HELLO]

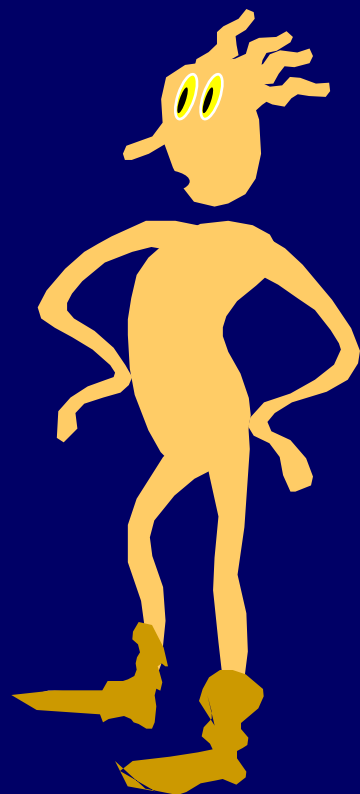
BUILD:
HELLO
Oracle

Are P and H equal?

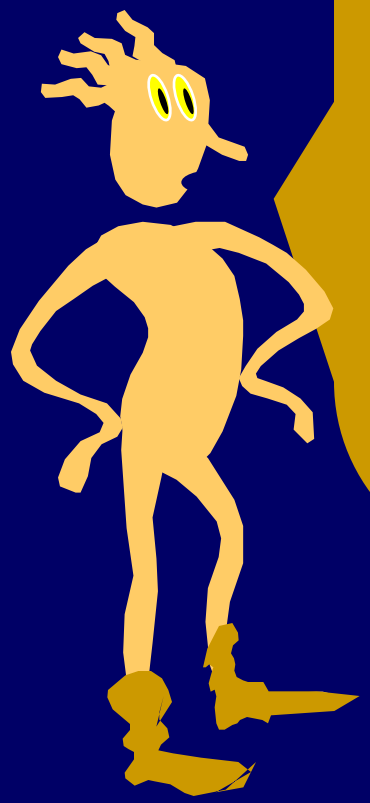
GIVEN:

EQUAL
Oracle

Halting with input,
Halting without input,
The “Hello World”
assignment, and
EQUAL are not
decidable.



What about problems that have no obvious relation to halting, or even to computation can encode the Halting Problem is non-obvious ways?



Diophantine equations

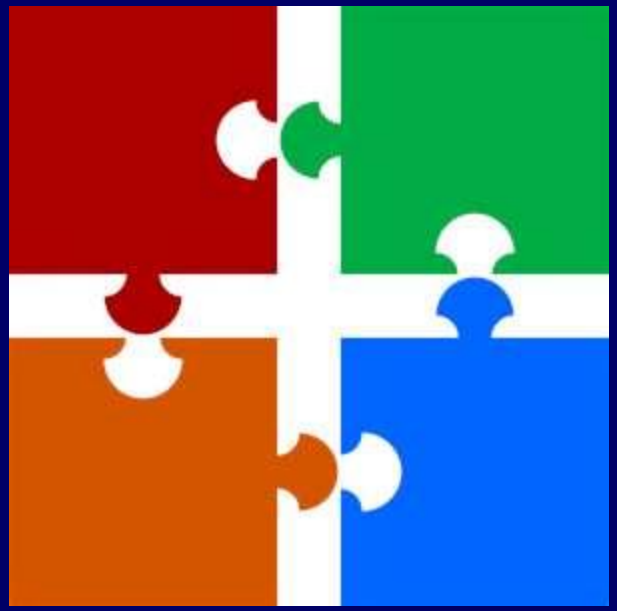
$$a^k + b^k = c^k$$

$$xy^2 - xz = p$$

Hilberts 10th problem was to find a solution to such equations.

Puzzle Pieces

Given a finite set of puzzle pieces, can you tile the plane (you are allowed to use each piece arbitrarily often)?



PHILOSOPHICAL INTERLUDE

CHURCH-TURING THESIS

Any well-defined procedure that can be grasped and performed by the human mind and pencil/paper, can be computed on a conventional digital computer with no bound on its memory.

The Church-Turing Thesis is NOT a theorem. It is a statement of belief about the universe we live in.

Your opinion will be influenced by your religious, scientific, and philosophical beliefs.

Empirical Intuition

No one has ever given a counter-example to the Church-Turing thesis. That is, no one has given a concrete example of something that humans can compute in a consistent and well defined way, that also can't be programmed on a computer.

The thesis is true.

Mechanical Intuition

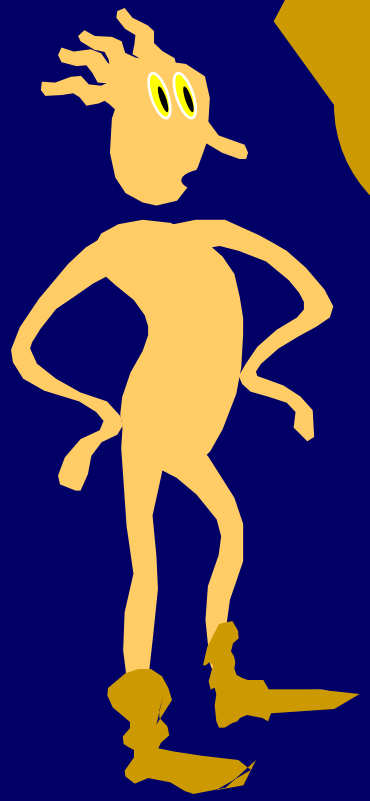
The brain is a machine. The components of the machine obey physical laws.

In principle, an entire brain can be simulated step by step on a digital computer. Thus, any thoughts of such a brain can be computed by a simulating computer. The thesis is true.

Spiritual Intuition

The mind consists of part matter and also part soul. Soul, by its very nature, cannot be reduced to physical laws. Thus, the action and thoughts of the brain cannot be simulated or reduced to simple components and rules. The thesis is false.

Do these theorems about the limits of computation tell us something about the limitations of human thought?



Self-Reference Puzzle

Write a program that prints its own code out as output.

No calls to the operating system, or to memory external to the program.

(You don't need to use a specific programming language, just your own "English pseudocode" will do.)